Robustness analysis and synthesis of a multi-PID controller based on an uncertain multimodel representation

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Abstract. This paper presents an effective method for robustness analysis and synthesis of a multi-PID controller for nonlinear systems where desirable robustness and performance properties must be maintained across a large range of operating conditions. The robustness analysis problem is solved using an uncertain multimodel of the original nonlinear system. The model of uncertainties used is an interval matrix modeled by a stochastic matrix which gives poor conservatism in the analysis of stability robustness. Moreover, the robust stability margin is interpreted as a smallest interval matrix that causes instability. This stability margin is evaluated using a random search algorithm. Simulation studies are used to demonstrate the effectiveness of the proposed method.

Keywords: PID controller; Multimodel; Multi-PID; Parametric uncertainties; Robust stability; Random search algorithm.

1 Introduction

The proportional-integral-derivative (PID) controller is the industrial standard for the control process. The popularity of the PID controller can be attributed partly to their performance which is satisfactory in many applications and partly to their functional simplicity, which allows engineers to operate them in a simple and straightforward manner. For these reasons, the majority of the controllers used in industry are of PI or PID type.

Most of real plant operate in a wide range of operating conditions, the robustness is then an important feature of the closed loop system. When this is the case, the controller has to be able to stabilize the system for all operating conditions. In this way, numerous progress has been made to improve the performances of the PI/PID control [3]. In particular, tuning methods based on optimization approach have recently received more attention in the literature, with the aim of ensuring good stability robustness of the controlled system [5, 2]. However these new methods are not very effective in the case of a strongly nonlinear system evolving on a large range of operating condition. Indeed, it is well known that a controller designed around a specific operating point may not be able to accomodate large variations in process dynamics. This is due mainly by the presence of system non linearities, which cause different dynamic behaviors from an operating point to another. This kind of problems can be solved by linearizing the system equations around several operating points and designing a linear controller for each region of operation. The resulting controllers are

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then combined by interpolation in order to obtain an appropriate controller for the original nonlinear system, this is the well known gain scheduling approach [6, 8]. This procedure is time consuming and expensive, but is well accepted and gives satisfactory results in many applications. Another similar approach facilitates the controllers interpolation through the use of validity or membership functions. By this method local controllers are selected as a function of the current state of the process [4, 1, 13]. These approaches leads naturally to the approximation of a nonlinear system by a family of linear systems correctly combined between them. This concept is not new and was first developed in an elegant manner by Takagi and Sugeno within the framework of the fuzzy set theory [10]. Takagi-Sugeno fuzzy models are non linear systems described by a set of if-then rules which gives local linear representation of an underlying system. Practically, such multi-linear modeling (multimodel) technique can be used to extend the well known linear controller design tools to complexe nonlinear systems [11, 7].

The main objective of this paper is to provide a simple and practical method for the evaluation of the stability robustness of a given PID controller in the context of parametric uncertainties. Contrary to existing solutions, the proposed method does not require the resolution of LMIs or BMIs [9, 15]. This is more interesting from a practical point of view, because in the context of robust control design, the LMI solution generally requires to simultaneously solve a number of convex inequalities which is exponential according to the number of parameters. Thus the LMI approach is computationally critical for a large number of uncertain parameters. The same is true for BMIs but in addition no efficient algorithm exists to solve BMIs.

In this work we propose to solve the robustness analysis problem by using a random search approach. To this end, a multimodel representation is used which is able to represent a given nonlinear process. The proposed multimodel takes into account parametric uncertainties, which result of the process linarisation or identification around a family of operating point, by the use of a stochastic matrix. On the basis of this uncertain multimodel, a random search algorithm is proposed to find an estimate of the largest parametric uncertainties before instability. Using this algorithm, a practical design method of a multi-PID controller is proposed which allows us to evaluate the robustness of the closed-loop system.

The paper is organized as follows. Section 2 presents the construction of the uncertain multimodel in order to obtain on the operating range a correct representation of the behavior of the original nonlinear system. Section 3 is devoted to the design of single PID controller that is able to stabilize the uncertain multimodel. Sufficient conditions for the robust asymptotic stability are given, which can be used to determine, for a given PID controller, the largest parametric uncertainties before instability (or equivalently the smallest parametric uncertainty that causes instability). In section 4 the single PID controller is generalized to a multi-PID controller and a design methodology is presented. Simulation studies are conducted in section 5, finally section 6 concludes this paper.

2 The multimodel used for the synthesis

Consider the class of nonlinear single-input single-output (SISO) plants expressed in the following form:

$$\left\{ \begin{array}{l} \dot{z} = f(z,u) \\ y = h(z) \end{array} \right. \tag{1}$$

where $z \in \mathbf{R}^{n_z}$ denote the state vector, $u \in \mathbf{R}$ is the control input, $y \in \mathbf{R}$ is the measured output, and f and h are smooth functions on $\mathbf{R}^{n_z}$ and $\mathbf{R}$, respectively. The design of an output feedback controller that stabilise the nonlinear system (1) remains relatively difficult. However, it is well known that this system can be correctly represented by an appropriate combination of linear local models.

Let $\mathcal{D}$ the desired operating domain of the underlying system. This domain can be divided into l local domains $\mathcal{D}_i$ where the system (1) can be represented by a local linear model.

Assumption 2.1. On each local domain $\mathcal{D}_i$, the system (1) can be described by the following local linear state space representation:

$$\left\{ \begin{array}{l} \dot{x}(t) = A_i x(t) + B_i u(t) \\ y(t) = Cx(t) \end{array} \right. \tag{2}$$

where $x \in \mathbf{R}^n$ denote the state vector, $u \in \mathbf{R}$ is the control input, $y \in \mathbf{R}$ is the measured output, A_i, B_i and C are constants matrices with appropriates dimensions. Note that it is always possible to put a given local model (strictly proper) in the form (2).

This local model can be obtained by an appropriate identification of the system around an operating point $y_0^i \in \mathcal{D}_i$, possibly by a local first-order plus dead time model or a second-order plus dead time model (see remark 2.1). In the case where de nonlinear process model (1) is known, this local representation can be obtained by linearisation via first order Taylor series expansion of the nonlinear functions f and h.

Remark 2.1. It is well known that a very large class of industrial process can be represented, around a given operating point, by a first-order plus dead time model $G(s) = ke^{-t_0 s}/(1+\tau s)$, or a second-order plus dead time model $G(s) = k\omega_0^2 e^{-t_0 s}/(s^2 + 2\zeta\omega_0 s + \omega_0^2)$. Using the approximation $e^{-t_0 s} \approx 1/(1 + \frac{t_0}{\alpha}s)^\beta$, where the constant β is choozing in order to obtain a good accuracy of the time delay, the transfer function of the process model is then given by $G(s) = k/(s^n + \cdots + a_1 s + a_0)$, with $n = \beta + 1$ (first order case), or $n = \beta + 2$ (second order case). It can be verified that (2) is a state space realisation of this transfer function.

However, the local model (2) is valid only around the operating point $y_0^i \in \mathcal{D}_i$, and is called the nominal local linear model of the domain $\mathcal{D}_i$. For $y \neq y_0^i$ and $y \in \mathcal{D}_i$, the corresponding dynamic behavior of the process can be very different of that of the nominal local linear model. In fact, the system matrices (A_i,B_i) varies on the domain $\mathcal{D}_i$. These variations can be seen as a parameters uncertainties. In order to take into account these uncertainties, it is necessary to consider an uncertain linear local model.

Assumption 2.2. On the domain $\mathcal{D}_i$, the system matrices (A_i,B_i) verifies the following inequalities :

$$\left\{ \begin{array}{l} \underline{A}_i \leqslant_e A_i \leqslant_e \bar{A}_i \\ \underline{B}_i \leqslant_e B_i \leqslant_e \bar{B}_i \end{array} \right. \tag{3}$$

where the matrices $\underline{A}_i$, $\bar{A}_i$, $\underline{B}_i$ and $\bar{B}_i$ are known bounds of the nominal matrices A_i and B_i respectively.

Using this assumption, the matrices A_i and B_i can be rewritten as follows:

$$\left\{ \begin{array}{l} A_i = A_i^0 + \Xi_A(t) \circledast A_i^1 \\ B_i = B_i^0 + \Xi_B(t) \circledast B_i^1 \end{array} \right. \text{ with:} \left\{ \begin{array}{l} |\,\Xi_A(k,q)(t)| \leqslant 1, \ \ |\,\Xi_B(k,q)(t)| \leqslant 1 \\ A_i^0 = \frac{1}{2}(\underline{A}_i + \bar{A}_i), \ \ A_i^1 = \frac{1}{2}(\bar{A}_i - \underline{A}_i) \\ B_i^0 = \frac{1}{2}(\underline{B}_i + \bar{B}_i), \ \ B_i^1 = \frac{1}{2}(\bar{B}_i - \underline{B}_i) \end{array} \right. \tag{4}$$

Remark 2.2. With this new formulation of uncertainties, the matrices A_i and B_i are not quelconque, for instance, each element of A_i is such that $A_i(k,q) \in [\underline{A}_i(k,q), \bar{A}_i(k,q)]$. This representation is then more realistic compared to bounded uncertainties which lead to very conservative results. Indeed, bounded uncertainties include matrices which never appear in the evolution of the real system on the domain $\mathcal{D}_i$.

The uncertain linear local model on the domain $\mathcal{D}_i$, is then written as follows:

$$\left\{ \begin{array}{l} \dot{x}(t) = (A_i^0 + \Xi_A(t) \circledast A_i^1)x(t) + (B_i^0 + \Xi_B(t) \circledast B_i^1)u(t) \\ y(t) = Cx(t), \quad y \in \mathcal{D}_i \end{array} \right. \quad i = 1,\ldots,l \tag{5}$$

We have thus a family of l local models allowing to represent validly the behavior of the system on each local areas. If $y \in \mathcal{D}_i$ then the local model number i describe correctly the system. This idea can be formalized in the following manner, let $\mu_i(y) > 0$ a function allowing to indicate the validity of the local model number i on the domain $\mathcal{D}_i$:

$$\mu_i(y) : \ \mathcal{D} = \cup_i \mathcal{D}_i \to [0,\, 1] \tag{6}$$

such that $\mu_i(y) \approx 1$ for $y \in \mathcal{D}_i$ and decreasing rapidly to zero beyond $\mathcal{D}_i$. The non linear process model (1) can then be approximated by interpolation of the linear uncertain local models (5):

$$\left\{ \begin{array}{l} \dot{x}(t) = \displaystyle\sum_{i=1}^{l} w_i(y) \left[\left(A_i^0 + \Xi_A(t) \circledast A_i^1\right) x(t) + \left(B_i^0 + \Xi_B(t) \circledast B_i^1\right) u(t) \right] \\ y(t) = Cx(t), \quad y \in \mathcal{D} = \cup_i \mathcal{D}_i \\ w_i(y) = \dfrac{\mu_i(y)}{\sum_{i=1}^{l} \mu_i(y)} \end{array} \right. \tag{7}$$

where $w_i(y)$ is the interpolation function which connect smoothly the local models together in order to form, on the domain $\mathcal{D}$, a global model of the non linear system (1).

Example 2.1. Consider the model of a stirred tank reactor :

$$\begin{aligned} \dot{C}_A = & \ \frac{q}{V}(C_{Af} - C_A) - k_0 C_A e^{-\frac{E}{RT}} \\ \dot{T} = & \ \frac{q}{V}(T_f - T) - \frac{\Delta H k_0}{\rho C_p} C_A e^{-\frac{E}{RT}} \\ & + \frac{\rho_c C_{pc}}{\rho C_p V} q_c (1 - e^{-\frac{h_A}{\rho_c C_{pc} q_c}})(T_{cf} - T) \end{aligned} \tag{8}$$

whose variables, parameters and nominal values are the same as defined in [2] and reproduced below.

Parameter	Notation	Value
Process flow rate	q	100 l/min
Feed concentration	C_{Af}	1 mol/l
Feed temperature	T_f	350 K
Coolant inlet temperature	T_{cf}	350 K
Reactor volume	V	100 l
Heat transfer coefficient	h_A	7×10^5 cal/min/K
Reaction rate constant	k_0	7.2×10^{10} min^{-1}
Activation energy term	E/R	1×10^4 K
Heat of reaction	ΔH	-2×10^5 cal/mol
Liquid densities	ρ, ρ_c	1×10^3 g/l
Specific heat	C_p, C_{pc}	1 cal/g/K

In this example, C_A is the measured output, q_c is the control variable and C_{Af} is the disturbance. Consider the operating range defined as $\mathcal{D} = \{(C_A, T, q_c) : C_A \in [0.06, 0.13]\}$, for the operating points $C_A^1 = 0.06$, $C_A^2 = 0.1$ and $C_A^3 = 0.13$, the corresponding nominal local linear models are

$$A_1 = \begin{bmatrix} -16.67 & -0.047 \\ 3133.33 & 7.42 \end{bmatrix}, A_2 = \begin{bmatrix} -10 & -0.047 \\ 1800 & 7.33 \end{bmatrix}, A_3 = \begin{bmatrix} -7.69 & -0.046 \\ 1338.46 & 7.19 \end{bmatrix}$$

$$B_1^T = [0 \quad -0.99],\ B_2^T = [0 \quad -0.88],\ B_3^T = [0 \quad -0.82]$$

$$T^1 = 449.47,\ q_c^1 = 89.03,\ T^2 = 438.54,\ q_c^2 = 103.41,\ T^3 = 432.92,\ q_c^3 = 110.03$$

For gaussian validity functions, the nominal multimodel is given by:

$$\begin{cases} \begin{bmatrix} \dot{C}_A(t) \\ \dot{T}(t) \end{bmatrix} = \sum_{i=1}^{3} w_i(C_A) \left(A_i \begin{bmatrix} C_A(t) - C_A^i \\ T(t) - T^i \end{bmatrix} + B_i(q_c(t) - q_c^i) \right) \\ w_i(C_A) = \frac{\mu_i(C_A)}{\sum_{j=1}^{3} \mu_j(C_A)}, \quad \mu_i = \exp\left[-\frac{1}{2} \left(\frac{C_A - C_A^i}{\sigma_i} \right)^2 \right] \end{cases}$$

where the parameters σ_i are chosen in order to cover the totality of the operating range (in this example $\sigma_i = 0.01$, $i = 1{,}3$). Comparison results are shown in figure 1 for successive step changes in the flow rate that varie between $q_c = 89.03 l/min$ and $q_c = 110.03 l/min$.

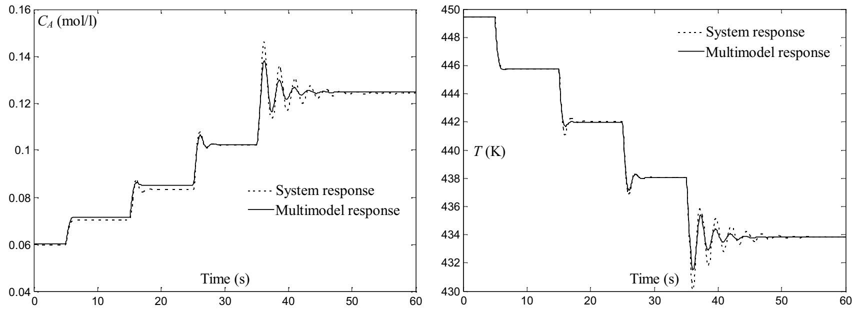

FIG. 1 – *Open-loop responses to successive step changes in the flow rate.*

One can see that in the whole operating range ($C_A \in [0.06 \quad 0.13]$), the multimodel is a good approximation of the nonlinear system but, of course, not represent exactly the real system. If in this nominal model is incorporated the proposed model of uncertainties, one defines in fact a set of model which include the evolution of real system. Thus the stabilisation of the uncertain multimodel implies the stabilization of the real system. This aspect is studied in the next section.

3 Robust stabilization of the uncertain multimodel

Consider a n^{th}-order multimodel with parametric uncertainties, which is described by the following state space equation:

$$\left\{ \begin{array}{l} \dot{x}(t) = \sum_{i=1}^{l} w_i(y) \left[\left(A_i^0 + \Xi_A(t) \circledast A_i^1 \right) x(t) + \left(B_i^0 + \Xi_B(t) \circledast B_i^1 \right) u(t) \right] \\ y(t) = Cx(t), \quad w_i(y) = \dfrac{\mu_i(y)}{\sum_{i=1}^{l} \mu_i(y)} \end{array} \right. \tag{9}$$

The objective is to design a PID controller for robust stabilisation of (9), the control law is in the following standard form:

$$\left\{ \begin{array}{l} \dot{\xi}_1(t) = -\frac{1}{\tau_d}\xi_1(t) + \frac{1}{\tau_d}\varepsilon(t) \\ u(t) = k_i \int_0^t \varepsilon(\tau)d\tau + \dfrac{k_d}{\tau_d}(\varepsilon(t) - \xi_1(t)) + k_p\varepsilon(t) \end{array} \right. \tag{10}$$

where $\varepsilon = r - y$ is the error, r the reference input and y the measured output. Note that this representation in the time domain, corresponds to the following transfert function $\frac{u(s)}{\varepsilon(s)} = k_p + \frac{k_i}{s} + \frac{k_d s}{1+\tau_d s}$, which is proper, this PID is then physically realisable. Let $\dot{\xi}_2(t) = r(t) - y(t)$, the control input is then written as follows:

$$\left\{\begin{array}{l} \dot{\xi}_1(t) = -\frac{1}{\tau_d}\xi_1(t) + \frac{1}{\tau_d}\varepsilon(t) \\ \dot{\xi}_2(t) = r(t) - Cz(t) \\ u(t) = Kx_a(t) + K_r r(t) \end{array}\right. \tag{11}$$

with $K = \left[-\left(\frac{k_d}{\tau_d} + k_p\right)C \quad -\frac{k_d}{\tau_d} \quad k_i\right]$, $K_r = \left(\frac{k_d}{\tau_d} + k_p\right)$, and $x_a^T = [x^T \quad \xi_1 \quad \xi_2]$. The closed-loop system of (9) and (10) is then

$$\left\{\begin{array}{l} \dot{x}_a(t) = \displaystyle\sum_{i=1}^{l} w_i(y)\left\{\left[A_c^i + \tilde{A}_c^i + \left(B_c^i + \tilde{B}_c^i\right)K\right]x_a(t) + B_r^i r(t)\right\} \\ \\ y(t) = [C \quad 0_{1\times 2}]x_a(t) \end{array}\right. \tag{12}$$

with

$$A_c^i = \left[\begin{array}{c|c} A_i^0 & 0_{n\times 2} \\ \hline \begin{bmatrix} -\frac{1}{\tau_d}C \\ -C \end{bmatrix} & \begin{bmatrix} -\frac{1}{\tau_d} & 0 \\ 0 & 0 \end{bmatrix} \end{array}\right], \quad \tilde{A}_c^i = \left[\begin{array}{c|c} \Xi_A(t) \circledast A_i^1 & 0_{n\times 2} \\ \hline 0_{2\times n} & 0_{2\times 2} \end{array}\right]$$

$$B_c^i = \left[\begin{array}{c} B_i^0 \\ \hline 0_{2\times 1} \end{array}\right], \tilde{B}_c^i = \left[\begin{array}{c} \Xi_B(t) \circledast B_i^1 \\ \hline 0_{2\times 1} \end{array}\right], B_r^i = \left(B_c^i + \tilde{B}_c^i\right)K_r + \left[\begin{array}{c} 0_{n\times 1} \\ \hline \begin{bmatrix} \frac{1}{\tau_d} \\ 1 \end{bmatrix} \end{array}\right] \tag{13}$$

The main result for the asymptotic stability of the closed-loop uncertain multi-model (12), is summarized in the following theorem.

Theorem 3.1. *If there exist a symmetric and positive definite matrix P, a matrix $K = \left[-\left(\frac{k_d}{\tau_d} + k_p\right)C \quad -\frac{k_d}{\tau_d} \quad k_i\right]$ and a real number α such that:*

$$\left\{\begin{array}{l} \forall\, \Xi_A(k,q)(t), \Xi_B(k,q)(t) \in [-\alpha, \alpha], \;\; \lambda_{max}\left[Q_i + \Delta_i^T P + P\Delta_i\right] < 0 \\ Q_i = \left(A_c^i + B_c^i K\right)^T P + P\left(A_c^i + B_c^i K\right) \\ \Delta_i = \left[\begin{array}{c|c} \Xi_A(t) \circledast A_i^1 & 0_{n\times 2} \\ \hline 0_{2\times n} & 0_{2\times 2} \end{array}\right] + \left[\begin{array}{c} \Xi_B(t) \circledast B_i^1 \\ \hline 0_{2\times 1} \end{array}\right] K \end{array}\right. \tag{14}$$

for $i = 1, \ldots, l$, then the origin of the closed-loop system (12) is an asymptotically stable equilibrium point, for all parametric uncertainties satisfying:

$$\left\{\begin{array}{l} A_i^0 - \alpha A_i^1 \leqslant_e A_i \leqslant_e A_i^0 + \alpha A_i^1 \\ B_i^0 - \alpha B_i^1 \leqslant_e B_i \leqslant_e B_i^0 + \alpha B_i^1 \end{array}\right. \quad i = 1, \ldots, l \tag{15}$$

Proof. For the simplicity of notation, define $\tilde{Q}_i = A_c^i + B_c^i K + \Delta_i$. Consider the Lyapunov function candidate $V(x_a) = x_a^T P x_a$, where P is a time invariant, symmetric and positive definite matrix. By substituting (12) with $r = 0$, into the time derivative of $V(x_a)$, $\dot{V}(x_a) = \dot{x}_a^T P x_a + x_a^T P \dot{x}_a$, we obtain

$$\begin{aligned} \dot{V}(x_a) = &\; x_a^T \left\{\textstyle\sum_{i=1}^{l} w_i(y)\tilde{Q}_i\right\}^T P x_a + x_a^T P \left\{\textstyle\sum_{i=1}^{l} w_i(y)\tilde{Q}_i\right\} x_a \\ = &\; x_a^T \left\{\textstyle\sum_{i=1}^{l} w_i(y)\left(\tilde{Q}_i^T P + P\tilde{Q}_i\right)\right\} x_a \\ = &\; x_a^T \left\{\textstyle\sum_{i=1}^{l} w_i(y)\left(Q_i + \Delta_i^T P + P\Delta_i\right)\right\} x_a \end{aligned}$$

Let $\lambda_{max}[.]$ the largest eigenvalue of the symmetric matrix [.], if

$$\lambda_{max}\left[Q_i + \Delta_i^T P + P\Delta_i\right] < 0$$

for $i = 1,\ldots,l$ and $\Xi_A(k,q)(t), \Xi_B(k,q)(t) \in [-\alpha,\ \alpha]$, then $\dot{V}(x_a) < 0$. The uncertain closed-loop multimodel is then asymptotically stable on the domain $\mathcal{D} = \cup_j \mathcal{D}_j$ for all uncertainties satisfying (15).

Theorem 3.1 cannot be used to compute directly the matrix gain K, however the following design procedure can be adopted for this task.

1. *Compute the matrix gain K for the worst-case local model. For this, one can use, for example, the method described in [14].*
2. *Verifies that the following inequality* $\max_i$ *Re* $\left\{\lambda[(A_c^i + B_c^i)K]\right\} < 0$ *is satisfied.*
3. *Search the largest number $\alpha = \alpha_{max}$ such that*

$$\max_{i,\,\alpha} \mathrm{Re}\left\{\lambda\left[A_c^i + \tilde{A}_c^i + \left(B_c^i + \tilde{B}_c^i\right)K\right]\right\} < 0 \tag{16}$$

   For $\alpha \leqslant \alpha_{max}$, the PID controller stabilise the multimodel for all uncertainties satisfying (15).

3.1 Robustness analysis

It is now necessary to give some developments concerning the step three of the procedure given above. Note that the value α_{max} can be seen as a robust stability margin, that is the smallest symmetric uncertainty that causes instability. Indeed, consider a sufficiently small $\alpha > 0$ such that the closed-loop system is stable. Next increase α until α_{max} so that the closed-loop system becomes unstable. Thus α_{max} is the robust stability margin. The determination of this stability margin is not an easy task because this problem is related to the stability of an interval matrix which is a NP-hard problem [16]. An interesting approach to solve this kind of problem is to use a random search algorithm. For this, consider, for given matrices P, K and a number α the following function:

$$\Psi(\Xi_A(t),\Xi_B(t)) = \begin{cases} 1 & \text{if } \max\limits_i \mathrm{Re}\left\{\lambda\left[A_c^i + \tilde{A}_c^i + \left(B_c^i + \tilde{B}_c^i\right)K\right]\right\} < 0 \\ 0 & \text{otherwise} \end{cases} \tag{17}$$

the main result to check the robust stability of the closed-loop uncertain multimodel (12), is summarized in the following theorem.

Theorem 3.2. *If the algorithm*

1. *Choose a number of iterations η, and a size of uncertainties α.*
2. *Generate the matrices Ξ_A and Ξ_B, with random uniformly distributed elements on the interval $[-\alpha,\ \alpha]$.*
3. *If $\Psi(\Xi_A,\Xi_B) = 0$ then stop.*
4. *If the number of iterations is incomplete go to step 2, otherwise stop.*

is stopped without to have $\Psi(\Xi_A,\Xi_B) = 0$ *after a number of iterations* η *such that* $\eta \geqslant \ln(\delta)/\ln(1-\epsilon)$*, then, with a confidence* $1-\delta$*, the closed-loop system can be declared stable for all system parameters satisfying (15), except possibly for those belonging to a set of measure no larger than* ϵ*.*

Proof. For a given probability of instability $\epsilon = \Pr\{\Psi = 0\}$, which is considered as an improbable and indesirable event, we want to determine the number of iterations in order to detect the apparition of this indesirable event with a given probability $1-\delta$. Consider a total number of η experiments (or iterations), the probability so that we have one closed-loop system instable after successive closed-loop systems stable is given by $\epsilon+\sum_{k=2}^{\eta}\epsilon(1-\epsilon)^{k-1}$. We want to determine the number of iterations in order to detect the apparition of an instable closed-loop system with a probability at least $1-\delta$. The problem is then to determine η such that $\epsilon+\sum_{k=2}^{\eta}\epsilon(1-\epsilon)^{k-1} \geqslant 1-\delta$, which gives:

$$\eta \geqslant \ln(\delta)/\ln(1-\epsilon) \tag{18}$$

Finally, if the probability of instability is "known" to be ϵ, then the algorithm will detect at least an unstable instance within $\eta \geqslant \ln(\delta)/\ln(1-\epsilon)$, with a probability larger than $1-\delta$. Now assume that the algorithm runs up to all η iterations without detecting instability. This means only that true probability is less than or equal to ϵ, but not necessarily that the system is stable in all cases. Indeed, let ϵ' be the true but unknown probability of instability. For η iterations, we have not detected unstable system, this imply that the true probability of detection of an unstable system after successive stable systems, is such that:

$$\epsilon' + \sum_{k=2}^{\eta}\epsilon'(1-\epsilon')^{k-1} \leqslant \epsilon + \sum_{k=2}^{\eta}\epsilon(1-\epsilon)^{k-1}$$

which imply that $\epsilon' \leqslant \epsilon$. Note that, for a probability of instability ϵ, we have:

$$1-\left(\epsilon + \sum_{k=2}^{\eta}\epsilon(1-\epsilon)^{k-1}\right) \leqslant \delta$$

which means that the largest probability of non-detection of an unstable system is δ, consequently the smallest probability of detection of stable systems is $1-\delta$. Finally, if after $\eta \geqslant \ln(\delta)/\ln(1-\epsilon)$, iterations, only stable plants are generated, it can be asserted, with a confidence $1-\delta$, that the system is stable for all system parameters satisfying (15), except possibly for those belonging to a set of measure no larger than ϵ.

Based on this result, an estimate $\hat{\alpha}_{max}$ of α_{max}, can be determined using the following random search algorithm.

Algorithm 3.1.

1. *For a given* $\epsilon \in (0,1)$ *and* $\delta \in (0,1)$*, choose a number of iterations* η *such that* $\eta \geqslant \ln(\delta)/\ln(1-\epsilon)$*, a lower bound* $\alpha_I = \alpha_{inf}$*, an upper bound* $\alpha_S = \alpha_{sup}$ *such that* $\alpha_{inf} \leqslant \alpha_{max} \leqslant \alpha_{sup}$*, and a precision* ρ*.*

2. *Compute* $\alpha = \frac{\alpha_I + \alpha_S}{2}$

3. *Generate* η *i.i.d samples* $\left(\Xi_A^{(1)}, \Xi_B^{(1)}\right), \cdots, \left(\Xi_A^{(\eta)}, \Xi_B^{(\eta)}\right)$*, with random uniformly distributed elements on the interval* $[-\alpha, \alpha]$.

4. *If* $\sum_{i=1}^{\eta} \Psi\left(\Xi_A^{(i)}, \Xi_B^{(i)}\right) < \eta$ *then* $\alpha_S = \alpha$ *goto step 2, otherwise* $\alpha_I = \alpha$.

5. *If* $\alpha_S - \alpha_I > 2\rho\alpha_I$ *goto step 2, otherwise stop.*

Indeed, we always have $\alpha_I \leqslant \alpha_{max} \leqslant \alpha_S$. Moreover, after N iterations, $\alpha_I - \alpha_S = 2^{-N}(\alpha_{sup} - \alpha_{inf})$. Thus, when the algorithm is stopped, $(\alpha_I + \alpha_S)/2$ is guaranteed to approximate α_{max} within a relative accuracy of ρ, that is $|(\alpha_I + \alpha_S)/2 - \alpha_{max}| \leqslant \rho\alpha_{max}$.

By this approach, the dynamic performance of the closed-loop system varie on each local domain $\mathcal{D}_i$ because we have only one controller. If we want to obtain constant performance on the global domain $\mathcal{D} = \cup_i \mathcal{D}_i$, it is necessary to adopt a multi-PID controller, this is the purpose of the following section.

4 The multi-PID controller approach

Constant performance on the global domain $\mathcal{D} = \cup_i \mathcal{D}_i$ can be obtained using the following multi-PID controller

$$\left\{\begin{array}{l} \dot{\xi}(t) = -\frac{1}{\tau_d}\xi_1(t) + \frac{1}{\tau_d}\varepsilon(t) \\ \\ u(t) = \displaystyle\sum_{i=1}^{l} v_i(y)\left[k_1^i \int_0^t \varepsilon(\tau)d\tau + \frac{k_2^i}{\tau_d}(\varepsilon(t) - \xi_1(t)) + k_3^i \varepsilon(t)\right] \\ \\ v_i(y) = \left\{\begin{array}{l} 1 \text{ if } y(t) \in \mathcal{D}_i \\ 0 \text{ if } y(t) \notin \mathcal{D}_i \end{array}\right. \end{array}\right. \tag{19}$$

where the domain $\mathcal{D}_i$ is defined as follows

$$\mathcal{D}_i = \{y : \mu_i(y) \geqslant \mu_j(y), \quad j = 1, \ldots, l, \quad j \neq i\}, \quad i = 1, \ldots, l \tag{20}$$

with $\mu_i(y)$ defined as in (6). In the relation (19), $\varepsilon = r - y$ is the error, r the reference input and y the measured output. Let $\dot{\xi}_2(t) = r(t) - y(t)$, the control input is then written as follows:

$$\left\{\begin{array}{l} \dot{\xi}_1(t) = -\frac{1}{\tau_d}\xi_1(t) + \frac{1}{\tau_d}\varepsilon(t) \\ \dot{\xi}_2(t) = r(t) - Cz(t) \\ u(t) = \displaystyle\sum_{i=1}^{l} v_i(y)\left(K^i x_a(t) + K_r^i r(t)\right) \\ v_i(y) = \left\{\begin{array}{l} 1 \text{ if } y(t) \in \mathcal{D}_i \\ 0 \text{ if } y(t) \notin \mathcal{D}_i \end{array}\right. \end{array}\right. \tag{21}$$

with $K^i = \left[-\left(\frac{k_2^i}{\tau_d} + k_3^i\right) C \quad -\frac{k_2^i}{\tau_d} \quad k_1^i\right]$, $K_r^i = \left(\frac{k_2^i}{\tau_d} + k_3^i\right)$, and $x_a^T = [x^T \quad \xi_1 \quad \xi_2]$. The closed-loop system of (9) and (19) is then

$$\left\{\begin{array}{l} \dot{x}_a(t) = \sum_{i=1}^{l} \sum_{j=1}^{l} w_i(y) v_j(y) \left\{ \left[A_c^i + \tilde{A}_c^i + \left(B_c^i + \tilde{B}_c^i\right) K^j\right] x_a(t) + B_r^{ij} r(t)\right\} \\ \\ y(t) = [C \ \ 0_{1\times 2}] x_a(t) \end{array}\right. \tag{22}$$

where A_c^i, $\tilde{A}_c^i$, B_c^i and $\tilde{B}_c^i$ are defined in (13), B_r^{ij} is the same as B_r^i but K_r becomes K_r^j (see relation (13)). The main result for the asymptotic stability of the closed-loop uncertain multimodel (22), is summarized in the following theorem.

Theorem 4.1. *If there exist a set of symmetric and positive definite matrices P_j, a set of matrices $K^j = \left[-\left(\frac{k_2^j}{\tau_d} + k_3^j\right) C \quad -\frac{k_2^j}{\tau_d} \quad k_1^j\right]$ and a set of numbers α^j, (with $j = 1, \ldots, l$) such that:*

$$\left\{\begin{array}{l} \forall\, \Xi_A(k,q)(t), \Xi_B(k,q)(t) \in [-\alpha^j, \alpha^j],\ \lambda_{max}\left[Q_j + \Delta_j^T P_j + P_j \Delta_j\right] < 0 \\ Q_j = \left(A_c^j + B_c^j K^j\right)^T P_j + P_j \left(A_c^j + B_c^j K^j\right) \\ \Delta_j = \left[\begin{array}{c|c} \Xi_A(t) \circledast A_j^1 & 0_{n\times 2} \\ \hline 0_{2\times n} & 0_{2\times 2} \end{array}\right] + \left[\begin{array}{c} \Xi_B(t) \circledast B_j^1 \\ \hline 0_{2\times 1} \end{array}\right] K^j \end{array}\right. \tag{23}$$

for $j = 1, \ldots, l$, then the origin of the closed-loop system (22) is an asymptotically stable equilibrium point, for all parametric uncertainties satisfying:

$$\left\{\begin{array}{l} A_j^0 - \alpha^j A_j^1 \leqslant_e A_j \leqslant_e A_j^0 + \alpha^j A_j^1 \\ B_j^0 - \alpha^j B_j^1 \leqslant_e B_j \leqslant_e B_j^0 + \alpha^j B_j^1 \end{array}\right. \quad j = 1, \ldots, l \tag{24}$$

Proof. Suppose that the system evolve on the domain $\mathcal{D}_j$, on this domain only two local models are active, the models of indice $j-1$ and j or j and $j+1$, thus on the local domain $\mathcal{D}_j$ the closed-loop multimodel can be written as follows

$$\dot{x}_a(t) = \sum_{i=j-1}^{j+1} w_i(y) \left\{ \left[A_c^i + \tilde{A}_c^i + \left(B_c^i + \tilde{B}_c^i\right) K^j\right] x_a(t) + B_r^{ij} r(t)\right\} \tag{25}$$

note that the interval matrix generated with this expression is larger than the real interval matrix, and thus gives more conservative results for the stability studies. Let us recall that, in reality, when the system evolve on the domain $\mathcal{D}_j$, the system matrices (A_j, B_j) is such that $\underline{A}_j \leqslant_e A_j \leqslant_e \bar{A}_j$ and $\underline{B}_j \leqslant_e B_j \leqslant_e \bar{B}_j$, thus the state matrix of the closed-loop system is such that $\underline{A}_j + \underline{B}_j K^j \leqslant_e A_j + B_j K^j \leqslant_e \bar{A}_j + \bar{B}_j K^j$. For a weak conservatism, it is then convenient to study the stability on the domain $\mathcal{D}_j$ for the system defined as follows

$$\dot{x}_a(t) = \left[A_c^j + \tilde{A}_c^j + \left(B_c^j + \tilde{B}_c^j\right) K^j\right] x_a(t) + B_r^{jj} r(t) \tag{26}$$

Consider the Lyapunov function candidate $V_j(x_a) = x_a^T P_j x_a$, where P_j is a time invariant, symmetric and positive definite matrix. By substituting (26) with $r = 0$, into the time derivative of $V_j(x_a)$, $\dot{V}_j(x_a) = \dot{x}_a^T P_j x_a + x_a^T P_j \dot{x}_a$, we obtain

$$\dot{V}_j(x_a) = x_a^T \left[Q_j + \Delta_j^T P_j + P_j \Delta_j \right] x_a$$

Let $\lambda_{max}[.]$ the largest eigenvalue of the symetric matrix [.], if the following condition is satisfied

$$\forall\, \Xi_A(k,q)(t), \Xi_B(k,q)(t) \in [-\alpha^j, \alpha^j],\ \ \lambda_{max}\left[Q_j + \Delta_j^T P_j + P_j \Delta_j \right] < 0$$

for $j = 1,\ldots,l$ then $\dot{V}_j(x) < 0$ $(j = 1\ldots,l)$. The uncertain closed-loop multimodel is then asymptotically stable on the domain $\mathcal{D} = \cup_j \mathcal{D}_j$ for all uncertainties satisfying (24).

Theorem 4.1 cannot be used to compute directly the set of feedback gains $(K^1,\ldots,K^l)$, however the following design procedure can be adopted.

1. *For each local domain $\mathcal{D}_j$, compute the matrix gain K^j for the nominal local model (see remark 4.1).*
2. *Using Algorithm 3.1, find for each local domain $\mathcal{D}_j$, the largest number $\alpha^j = \alpha^j_{max}$ such that*

$$\max_{\alpha^j} \mathrm{Re}\left\{ \lambda\left[A_c^j + \tilde{A}_c^j + \left(B_c^j + \tilde{B}_c^j \right) K^j \right] \right\} < 0 \qquad (27)$$

Then, for all $\alpha^j \leqslant \alpha^j_{max}$, the multi-PID controller stabilise the uncertain multimodel for all uncertainties satisfying (24).

Remark 4.1. For low order local nominal models ($n \leqslant 2$), the design of the PID controller can be easily done by a classical poles placement. For high order nominal models ($n > 3$), the step 1 of the design procedure can be realised by using, for instance, the method presented in appendix.

5 Simulation results

Consider the model of a stirred tank reactor given in the example 2.1. For the operating range $\mathcal{D} = \{(C_A, T, q_c) : C_A \in [0.06,\, 0.13]\}$, we have, for example, the following local domains $D_1 = \{(C_A, T, q_c) : C_A \in [0.06,\, 0.080]\}$, $\mathcal{D}_2 = \{(C_A, T, q_c) : C_A \in [0.08,\, 0.115]\}$, $\mathcal{D}_3 = \{(C_A, T, q_c) : C_A \in [0.115,\, 0.13]\}$. On the domain $\mathcal{D}_i$, the matrices A_i and B_i $(i = 1,\ldots,3)$ can be written in the form $A_i = A_i^0 + \Xi_A(t) \circledast A_i^1$ and $B_i = B_i^0 + \Xi_B(t) \circledast B_i^1$, with:

$$A_1^0 = \begin{bmatrix} -14.6 & -0.047 \\ 2716.7 & 7.4 \end{bmatrix} \quad B_1^0 = \begin{bmatrix} 0 \\ -0.96 \end{bmatrix} \quad A_1^1 = \begin{bmatrix} 2.1 & 0 \\ 416.7 & 0.015 \end{bmatrix} \quad B_1^1 = \begin{bmatrix} 0 \\ 0.032 \end{bmatrix}$$

$$A_2^0 = \begin{bmatrix} -10.6 & -0.047 \\ 1919.6 & 7.3 \end{bmatrix} \quad B_2^0 = \begin{bmatrix} 0 \\ -0.89 \end{bmatrix} \quad A_2^1 = \begin{bmatrix} 1.9 & 0 \\ 380.4 & 0.065 \end{bmatrix} \quad B_2^1 = \begin{bmatrix} 0 \\ 0.041 \end{bmatrix}$$

$$A_3^0 = \begin{bmatrix} -8.2 & -0.047 \\ 1438.8 & 7.2 \end{bmatrix} \quad B_3^0 = \begin{bmatrix} 0 \\ -0.83 \end{bmatrix} \quad A_3^1 = \begin{bmatrix} 0.5 & 0 \\ 100.3 & 0.035 \end{bmatrix} \quad B_3^1 = \begin{bmatrix} 0 \\ 0.005 \end{bmatrix}$$

$$\forall t,k,q, \quad \Xi_A(k,q)(t) \in [-1,\, 1] \quad \text{and} \quad \Xi_B(k,q)(t) \in [-1,\, 1]$$

The controller parameters are determined by a classical poles placement in order to obtain a second order dominant mode with a first overshoot of 15% and a settling time of 1.5s. The parameters are as follows:

$$
\begin{array}{lllll}
\forall y \in \mathcal{D}_1, & k_1^1 = 1642 & k_2^1 = 62.06 & k_3^1 = 303.49 & \tau_d = 0.01 \\
\forall y \in \mathcal{D}_2, & k_1^2 = 1771 & k_2^2 = 160.17 & k_3^2 = 489.92 & \tau_d = 0.01 \\
\forall y \in \mathcal{D}_3, & k_1^3 = 1899 & k_2^3 = 230.71 & k_3^3 = 634.44 & \tau_d = 0.01
\end{array}
$$

On each domain, the robust stability margin, evaluated for $\epsilon = 0.005$ and $\delta = 0.005$, is $\alpha_{max}^1 = 0.79$, $\alpha_{max}^2 = 0.85$ and $\alpha_{max}^3 = 3.31$ respectively. Figure 2 shows the robustness analysis for each domain, and the closed-loop response for successive step changes in the effluent concentration C_A that varies between 0.06 and 0.14. It can be observed that in the whole operating regime, the dynamical performances obtained are similar in each sub-domain.

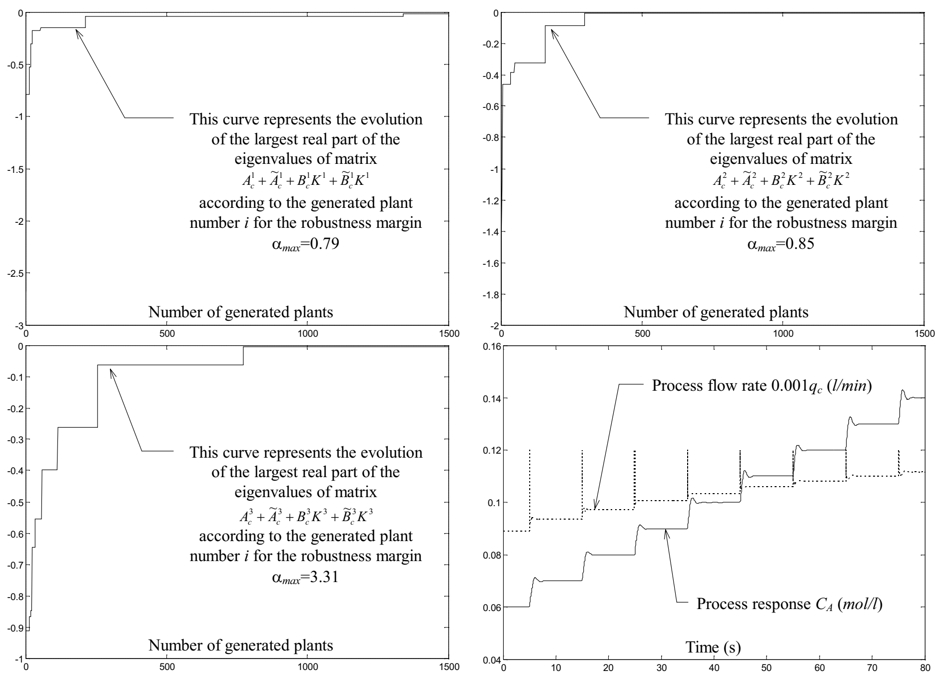

FIG. 2 – *Robustness analysis and closed-loop response to successive step changes in the effluent concentration.*

6 Conclusion

In this paper an effective method for robustness analysis and synthesis of a multi-PID controller for nonlinear systems was developed via uncertain multi-model approach. Simulation studies was used to demonstrate the effectiveness of the proposed method. The main results obtained in this paper can be easily generalized for multivariable PID controllers.

Appendix

A1. Robust design of a multi-PID controller for models of high degree

Let $L^i(s) = R^i(s)G^i(s)$ be the open-loop transfer function for the nominal local model number i. The transfer function of the local PID controller $R^i(s)$ and the transfer function of the local nominal model $G^i(s)$ are defined as follows:

$$\begin{array}{l} G^i(s) = C(sI - A_i)^{-1}B_i = \frac{b_m^i s^m + \cdots b_1^i s + b_0^i}{s^n + a_{n-1}^1 + \cdots a_1^i s + a_0^i} \\ R^i(s) = \frac{k_1^i}{s} + k_2^i s + k_3^i \end{array} \tag{28}$$

The consequences of the uncertainties on the parameters of the system are an uncertainty on the static gain (i.e. b_0^i/a_0^i) and an uncertainty on the dynamical behaviour of the system (i.e. the location of its poles and zeros). The objective is then to find the local PID controller parameters (k_1^i, k_2^i, k_3^i), so that the closed-loop system is not too sensitive to the model uncertainty and to have acceptable dynamical performance. The sensitivity of the closed-loop system to the uncertainty on the static gain can be reduced by an appropriate gain margin A_m. By definition on the gain margin we must have

$$L^i(\mathrm{j}\omega_\pi) = R^i(\mathrm{j}\omega_\pi)G^i(\mathrm{j}\omega_\pi) = -\frac{1}{A_m} \tag{29}$$

where ω_π is the phase crossover frequencie of the loop, and $\mathrm{j} = \sqrt{-1}$. Relation (29) can be rewritten as follows:

$$\left\{ \begin{array}{l} \alpha_i(\omega_\pi)k_3^i - \beta_i(\omega_\pi)\left(k_2^i\omega_\pi - \frac{k_1^i}{\omega_\pi}\right) = -\frac{1}{A_m} \\ \beta_i(\omega_\pi)k_3^i + \alpha_i(\omega_\pi)\left(k_2^i\omega_\pi - \frac{k_1^i}{\omega_\pi}\right) = 0 \end{array} \right. \tag{30}$$

where $\alpha_i(\omega)$ and $\beta_i(\omega)$ are the real part and imaginary part of $G^i(\mathrm{j}\omega)$ respectively. The solution of (30) is given by:

$$k_3^i = -\frac{\alpha_i(\omega_\pi)}{(\alpha_i(\omega_\pi)^2 + \beta_i(\omega_\pi)^2)A_m}, \quad k_1^i = k_2^i\omega_\pi^2 - \frac{\beta_i(\omega_\pi)\omega_\pi}{(\alpha_i(\omega_\pi)^2 + \beta_i(\omega_\pi)^2)A_m} \tag{31}$$

Thus for a given gain margin A_m and a given ω_π, one can compute the proportional gain k_3^i and the relation between the integral gain k_1^i and the derivative gain k_2^i. Now we want to find k_2^i so that the closed-loop system is less sensitive as possible to the uncertainty on the dynamical behaviour and to obtain a good transient response and a good rejection on the load disturbance. As shown in [14] these objectives can be reached by maximising the shortest distance from the Nyquist curve of the open-loop transfer function (i.e. $L^i(\mathrm{j}\omega)$) to the critical point -1. Thus the derivative gain k_2^i is determined such that:

$$\max_{k_2^i} \min_{\omega} |1 + L^i(\mathrm{j}\omega)| \tag{32}$$

Finally, given the gain margin A_m and the frequency ω_π, the determination of the controller parameters (k_1^i, k_2^i k_3^i) is formulated as the following optimisation problem:

$$\begin{cases} \max_{k_2^i > 0} \min_{\omega} |1 + L^i(\mathrm{j}\omega)| \\ L^i(\mathrm{j}\omega) = R^i(\mathrm{j}\omega) G^i(\mathrm{j}\omega) = \left(k_3^i + \mathrm{j}\left(k_2^i \omega - \frac{k_1^i}{\omega}\right)\right)(\alpha_i(\omega) + \mathrm{j}\beta_i(\omega)) \\ k_3^i = -\dfrac{\alpha_i(\omega_\pi)}{(\alpha_i(\omega_\pi)^2 + \beta_i(\omega_\pi)^2) A_m} \\ k_1^i = k_2^i \omega_\pi^2 - \dfrac{\beta_i(\omega_\pi)\omega_\pi}{(\alpha_i(\omega_\pi)^2 + \beta_i(\omega_\pi)^2) A_m} \end{cases} \tag{33}$$

which is numerically easy to solve. There is no difficulty to consider the PID controller with filtered derivative action (i.e. using $k_2^i s/(1+\tau_d s)$ instead of $k_2^i s$). Choosing for example $\tau_d \ll \omega_\pi^{-1}$, has a minimal effect on the shape of $L^i(\mathrm{j}\omega)$ for $\omega \leqslant \omega_\pi$. Alternatively, the PID can be assumed of the form $R^i(s) = \frac{k_1^i}{s} + \frac{k_2^i s}{1+\tau_d s} + k_3^i$, with τ_d fixed before solving the optimisation problem (33). In this case, we have:

$$L^i(s) = R^i(s) G^i(s) = \left((k_3^i + k_1^i \tau_d) + \frac{k_1^i}{s} + (k_2^i + k_3^i \tau_d) s\right) \frac{G^i(s)}{1 + \tau_d s} \tag{34}$$

The design procedure can then be applied by redefining the transfer function of the system as $G_a^i(s) = \frac{1}{1+\tau_d s} G^i(s)$, and considering the new PID parameters:

$$(k_1^i)' = k_1^i, \quad (k_2^i)' = k_2^i + k_3^i \tau_d, \quad (k_3^i)' = k_3^i + k_1^i \tau_d \tag{35}$$

Note that, contrary to existing PID design methods, the proposed approach does not use any simplification on the model used for the representation of the local behaviour of the considered system.

A2. Nomenclature

α	a positive real scalar
α_{max}	robust stability margin
$\alpha_i(\omega)$, $\beta_i(\omega)$	real and imaginary part of $G^i(\mathrm{j}\omega)$, respectively
δ	probability of non detection of an unstable system
ϵ	probability of instability
ε	difference between the reference input and the output $\varepsilon = r - y$
η	number of iterations necessary to detect an unstable system
$\lambda[.]$	set of eigenvalues of the matrix [.]
$\lambda_{max}[.]$	largest eigenvalue of the symmetric matrix [.]
$\mu_i(.)$	validity function of the local model number i on the domain $\mathcal{D}_i$
ξ	state vector of the PID controller $\xi = [\xi_1\ \xi_2]^T$
$\Xi_A(t)$, $\Xi_B(t)$	stochastic matrices modelling the uncertainties on matrices A_i and B_i
$\Xi_A^{(i)}$, $\Xi_B^{(i)}$	the sample, number i, of the stochastic matrices $\Xi_A(t)$ and $\Xi_B(t)$
τ_d	time constant of the filtered derivative action of the PID
ω	frequency (rad/sec)
ω_π	phase crossover frequency
$A(i,j)$	element in row i and column j of the matrix A
A_i, B_i, C	state matrices of the local linear model on the domain $\mathcal{D}_i$
$\bar{A}_i$, $\bar{B}_i$	upper bounds of the matrices A_i and B_i respectively
$\underline{A}_i$, $\underline{B}_i$	lower bounds of the matrices A_i and B_i respectively
A_i^0, B_i^0	"medium matrices" $A_i^0 = \frac{1}{2}(\underline{A}_i + \bar{A}_i)$, $B_i^0 = \frac{1}{2}(\underline{B}_i + \bar{B}_i)$
A_i^1, B_i^1	"deviation matrices" $A_i^1 = \frac{1}{2}(\bar{A}_i - \underline{A}_i)$, $B_i^1 = \frac{1}{2}(\bar{B}_i - \underline{B}_i)$
A_m	gain margin and phase crossover frequency
$\mathcal{D}$	operating domain of the nonlinear system
$\mathcal{D}_i$	sub-operating domain number i of $\mathcal{D}$
$f(.)$	state function of the nonlinear system
$G^i(s)$	transfer function of the nominal local model number i
$h(.)$	output function of the nonlinear system
i, j, k, q	indices
k_i, k_d, k_p	tuning parameters of the single PID controller
k_1^i, k_2^i, k_3^i	tuning parameters of the local PID controller number i
l	number of local models
n	order of the model
$R^i(s)$	Transfer function of the local PID controller number i
r	reference input
s	Laplace variable
u	input variable of the system
$V(.)$, $V_j(.)$	Lyapunov functions
$v_i(.)$	interpolation function of the multi-PID controller
$w_i(.)$	interpolation function of the multimodel
x_a	augmented state vector $x_a = [x^T\ \xi^T]^T$
x, y	state vector and output of the system, respectively
$\Pr\{.\}$	probability of the event $\{.\}$
$\mathrm{Re}\{.\}$, $\mathrm{Im}\{.\}$	Real and Imaginary part respectively of the complex number $\{.\}$
$M_1 \circledast M_2$	product element-by-element of the matrices M_1 and M_2
$M_1 \leqslant_e M_2$	inequality element-by-element of the matrices M_1 and M_2
$(.)^T$	transpose of the vector or the matrix (.)

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