A probabilistic approach to find a robust output feedback controller

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Abstract - This paper presents a simple but effective method for finding a robust output feedback controller via random search algorithm. The convergence of this algorithm can be guaranteed. Moreover, the probability to find a solution as well as the number of random trials can be estimated. The robustness of the closed loop system is improved by the minimization of a given cost function reflecting the performance of the controller for a set of plants. Simulation studies are used to demonstrate the effectiveness of the proposed method.

Keywords— Output feedback; Pole placement; Random search algorithm; Robustness; Condition number; Cost function; NP-hard problem.

I. INTRODUCTION

It is well known that the performances of feedback control systems are mainly determined by the locality of their closed-loop poles, it follows that a natural design approach to find a static output feedback is by means of pole placement. Compared to the pole placement via state feedback, the same problem via output feedback is more complexe. In fact, the static output feedback problem in the case where the feedback gain are constrained to lie in some intervals is NP-hard [BLO 97a], [BLO 97b], [FU 04]. In other words, if a deterministic algorithm for solving the static output feedback problem is derived, it is an exponential-time algorithm, this implies that moderately large problems are computationally intractable.

Starting from this negative result numerous progress has been made which modifies our notion of solving a given problem. In particular, randomized algorithms have recently received more attention in the literature. Indeed, for a randomized algorithm it is not required that it work all of the time but most of the time, in return, this kind of algorithms runs in polynomial-time [VID 01]. The idea of using a random algorithm to solve a complex problem is not new, and was first proposed, in the domain of automatic control, by Matyas [MAT 65], further developments can be found in [BAB 89] and references therein, [GOL 89], [KHA 96], [TEM 97], [VID 97], [VID 01], [KOL 01], [ABD 02].

The main objective of this paper is to present a new random approach to find a static output feedback for uncertain linear systems, which is simple and easy to use. In the proposed method, the probability to find a solution as well as the number of random trials can be evaluated. The robustness of the closed loop system is improved by the minimization of a given cost function reflecting the performance of the controller for a set of plants.

The paper is organized as follows. In section 2, the problem of static output feedback is formulated. Section 3 shows that this problem can be solved by an appropriate random search algorithm. The robustness issue is discussed in section 5, and section 6 presents various simulations results to demonstrate the effectiveness of this approach. Finally, section 6 concludes this paper.

II. PROBLEM STATEMENT

Consider a multivariable linear dynamic system described by

\[
\begin{align*}
\dot{x}(t) &= Ax(t) + Bu(t) \\
y(t) &= Cx(t)
\end{align*}
\]  

(1)

where \(x \in \mathbb{R}^n\), \(u \in \mathbb{R}^m\) and \(y \in \mathbb{R}^p\) represent the state, input and output vectors, respectively, \(A \in \mathbb{R}^{n \times n}\), \(B \in \mathbb{R}^{n \times m}\) and \(C \in \mathbb{R}^{p \times n}\) are known constant matrices. As usual, it is assumed that rank\([B] = m\) and rank\([C] = p\).

By applying a constant output feedback law

\[u(t) = r(t) + Ky(t)\]  

(2)

to (1), the closed-loop system is given as

\[
\begin{align*}
\dot{x}(t) &= (A + BK)\, x(t) + Br(t) \\
y(t) &= C x(t)
\end{align*}
\]  

(3)

where \(r \in \mathbb{R}^m\) is the reference input vector and \(K \in \mathbb{R}^{m \times p}\) the output feedback gain matrix. It was established that under the condition of \((A, B)\) controllable and \((C, A)\) observable and that \(m + p > n\) (see [DAV 75], [KIM 77]) or \(mp > n\) (e.g. see [WAN 92]), it exists a feedback gain matrix \(K\) such that \(\lambda(A + BK) = \Lambda\), where \(\Lambda\) is a given set of real and self-conjugate complex numbers, \(\Lambda = \{\lambda_1, \lambda_2, \ldots, \lambda_n\}\) as the desired poles of the closed-loop system and \(\lambda(M)\) is the spectrum of the square matrix \(M\). The exact pole placement problem for output feedback is to find such a \(K\). However, it is commonly recognized that in practical applications, the poles assigned are not required to be exactly the same as those specified. This is
because the closed-loop system with poles approximately close to the desired one will possess similar desired behavior [CHU 93]. In fact, from a practical point of view, it is sufficient to consider the pole placement in a specified stable region $\mathcal{D}$ of the complex plane.

As shown in [BLO 97a], the problem of finding an output feedback matrix $K$ such that $\overline{k}_{ij} \leq k_{ij} \leq \overline{k}_{ij}$ for all $i, j$, and such that $A + BKC$ is a stable matrix is NP-hard. In the next section is proposed a random search algorithm in order to find a constrained output feedback matrix $K$ (i.e. such that $\overline{k}_{ij} \leq k_{ij} \leq \overline{k}_{ij}$ for all $i, j$) such that $\lambda(A + BKC) \subset \mathcal{D}$, where $\mathcal{D}$ is a specified region of the complex plane determined in order to obtain a desired behavior. This problem is also NP-hard.

III. RANDOM SEARCH ALGORITHM APPROACH

The main result for finding a constrained output feedback matrix for the system (1) is summarized in the following theorem.

**Theorem 1.** If there exists an output feedback matrix $K$ such that $\overline{k}_{ij} \leq k_{ij} \leq \overline{k}_{ij}$ for all $i, j$, and such that $\lambda(A + BKC) \subset \mathcal{D}$, with $\mathcal{D} \subset \mathbb{C}^-$, then the algorithm

1) Generate a $m \times p$ matrix $K$ with random uniformly distributed elements $k_{ij}$ on the intervals $[\overline{k}_{ij}, \overline{k}_{ij}]$ for all $i, j$.

2) If $\lambda(A + BKC) \not\subset \mathcal{D}$, go to step 1, otherwise stop.

**Proof.** Let $\mathcal{K}$ be the set of $\mathcal{D}$-stabilizing output feedback matrices $K$ defined by

$$\mathcal{K} = \{ K \in \mathbb{R}^{m \times p} : \lambda(A + BKC) \subset \mathcal{D}, \quad \overline{k}_{ij} \leq k_{ij} \leq \overline{k}_{ij} \forall i, j \}$$

(4)

Let us consider $n$ iterations of the algorithm, the probability so that $K \not\in \mathcal{K}$ is given by the binomial probability distribution

$$P\{K \not\in \mathcal{K}\} = \left[ \frac{n!}{r!(n-r)!} \right] \xi^r (1 - \xi)^{n-r} r = 0 \ldots n$$

(5)

where $r$ is the number success (i.e. the number of times that $K \in \mathcal{K}$) and $\xi$ the probability of elementary success. For $\xi > 0$ it is clear that $\lim_{n \to \infty} (1 - \xi)^n = 0$ the algorithm converge then certainly to a solution.

**Remark 1.** The number of iterations necessary to obtain a solution with a confidence at least to $1 - \delta$ is given by:

$$n \geq \frac{\ln(\delta)}{\ln(1 - \xi)}$$

(6)

Indeed, from (5) we want to have $(1 - \xi)^n \leq \delta$ which gives $n \geq \ln(\delta) / \ln(1 - \xi)$. The probability $\xi$ can be estimated as relative frequency $\xi_N = N_s / N$, with $N$ is the total number of samples and $N_s$ the number of samples such that $\lambda(A + BKC) \subset \mathcal{D}$. The problem is to determine the number of samples $N$ in order to obtain a reliable probabilistic estimate. More precisely, given the accuracy $\epsilon \in [0, 1]$ and the confidence $\delta \in [0, 1]$, the minimum of samples $N$ which guarantees that $P_r \{ |\xi - \xi_N| \leq \epsilon \} \geq 1 - \delta$ is given by the Chernoff bound [CHE 52]. $N \geq \ln(2) / (2\epsilon^2)$. Thus, the probability $\xi$ can be estimated using the following algorithm.

1) Choose a number of iterations $N$ such that $N \geq \ln(2) / (2\epsilon^2)$.

2) Generate a $m \times p$ matrix $K$ with random uniformly distributed elements $k_{ij}$ on the intervals $[\overline{k}_{ij}, \overline{k}_{ij}]$ for all $i, j$.

3) If $\lambda(A + BKC) \subset \mathcal{D}$ then $N_s = N_s + 1$.

4) If the number of iterations is incomplete go to step 2, otherwise stop.

The estimation of the probability $\xi$ is then given by $N_s / N$.

IV. ROBUSTNESS ISSUE

In this section, our objective is to find an output feedback controller such that the closed-loop system remains stable for a large variety of plants. If there exists an uncertainty term $\Delta$ in the system matrix $A$, according to theorem 6 in [KAU 85], the closed-loop state matrix $A + \Delta + BKC$ is of Hurwitz if

$$\|\Delta\|_2 < \min_i \text{Re}(-\lambda_i) / \kappa_2(T)$$

(7)

where $\|\Delta\|_2$ is the 2-norm or spectral norm of $\Delta$, $\lambda_i (i = 1, 2, \ldots, n)$ are eigenvalues of $A + BKC$, $\kappa_2(T)$ is the spectral condition number of $T$, that is $\kappa_2(T) = \|T\|_2\|T^{-1}\|_2$, and $T$ is the eigenvector matrix of $A + BKC$. From inequality (7) one can see that a smaller $\kappa_2(T)$ gives a largest bound of $\|\Delta\|_2$ and thus increase the set of plants which can be stabilized. Hence the robustness of the closed loop system can be improved by solving the following optimization problem

$$\min_j \|T\|_2\|T^{-1}\|_2$$

subject to $K \in \mathcal{K}$.

(8)

A sub-optimal solution of this optimization problem can be found using the theorem 2.

**Theorem 2.** For a given level of performance if there exists a nonempty set of solutions $\mathcal{K}_\gamma = \{ K \in \mathcal{K} : J(K) \leq \gamma \}$, then the random optimization algorithm:
1) Select an initial output feedback matrix $K \in \mathcal{K}$, and a domain of exploration $[-d, d], d > 0$. 

2) Generate a $m \times p$ matrix $\Delta K$ with random uniformly distributed elements $\Delta k_{ij}$ on the interval $[-d, d]$ for $i, j$, such that $K + \Delta K \in \mathcal{K}$. 

3) If $J(K + \Delta K) < J(K)$ let $K = K + \Delta K$. 

4) If $J(K) > \gamma$, go to step 2, otherwise stop. 

converges certainly to a solution $K \in \mathcal{K}_\gamma$. 

**Proof.** Consider an initial matrix $K \in \mathcal{K}$ for which $J(K) > \gamma$. By assumption there exists $\Delta K$, with $K + \Delta K \in \mathcal{K}$, such that $J(K + \Delta K) < J(K)$. Consider $n$ iterations of the algorithm, the probability so that $J(K + \Delta K) > J(K)$ is given by $(1 - \xi)^n$ (see the proof of theorem 3.1), where $\xi > 0$ is the probability of success that is $Pr\{J(K + \Delta K) < J(K)\}$. It is clear that $\lim_{n \to \infty} (1 - \xi)^n = 0$, then repeating the step 2-3-4 we find finally $\Delta K$ such that $J(K + \Delta K) < J(K)$. If $J(K + \Delta K) \leq \gamma$ then $K + \Delta K \in \mathcal{K}_\gamma$, if not, we consider $K + \Delta K$ as a new initial matrix and repeating the reasoning above we see that the algorithm converge to an element of $\mathcal{K}_\gamma$ which is a suboptimal solution. Obviously, the optimal solution is given by the smallest level of performance $\gamma_{\text{min}}$ which is unknown.

V. SIMULATION RESULTS 

In this section various numerical examples are presented to illustrate the validity of the proposed approach. 

A. Example 1. 

Consider a 4-state, 2-input, 3-output aircraft example [YAN 94] given by 

$$
A = \begin{bmatrix}
-0.037 & 0.0123 & 0.00055 & -1.0 \\
0 & 0 & 1.0 & 0 \\
-0.37 & 0 & -0.23 & 0.0618 \\
1.25 & 0 & 0.016 & -0.0457 \\
\end{bmatrix}
$$

$$
B = \begin{bmatrix}
0.00084 & 0.000236 \\
0 & 0 \\
0.08 & 0.804 \\
-0.0862 & -0.0665 \\
\end{bmatrix}
$$

$$
C = \begin{bmatrix}
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
\end{bmatrix}
$$

with its open loop poles at $-0.0105, -0.2099, -0.0507 \pm 1.1168j$. We want to find an output feedback controller $K$, 

### Table I 

<table>
<thead>
<tr>
<th>$n$</th>
<th>Matrix gain and closed-loop poles</th>
<th>$\kappa_2$</th>
</tr>
</thead>
</table>
| 688 | $K = \begin{bmatrix}
0.2327 & 7.8452 & 3.5904 \\
-8.4385 & -6.6849 & 4.4334 \\
\end{bmatrix}$ | 12.5683 |
| 1621 | $K = \begin{bmatrix}
7.7624 & 8.0917 & 4.0072 \\
-8.8333 & -6.9531 & 4.4334 \\
\end{bmatrix}$ | 46.7701 |
| 403 | $K = \begin{bmatrix}
0.9900 & 2.1420 & 6.5890 \\
-1.0529 & -3.3637 & 2.7482 \\
\end{bmatrix}$ | 8.0678 |
| 175 | $K = \begin{bmatrix}
0.8021 & 0.6055 & 3.5335 \\
-0.7606 & -0.0396 & 3.0155 \\
\end{bmatrix}$ | 19.8294 |
| 1501 | $K = \begin{bmatrix}
-0.5261 & 0.2432 & 7.2889 \\
-1.4500 & -2.6392 & 0.9264 \\
\end{bmatrix}$ | 8.6107 |

### Table II 

<table>
<thead>
<tr>
<th>$n$</th>
<th>Matrix gain and closed-loop poles</th>
<th>$\kappa_2$</th>
</tr>
</thead>
</table>
| 75910 | $K = \begin{bmatrix}
1.5474 & 7.7801 & 8.5192 \\
-1.6833 & -3.7338 & 0.4161 \\
\end{bmatrix}$ | 5.2715 |

with $|k_{ij}| \leq 10 \forall i, j$, such that the closed-loop poles are in the region defined by $\mathcal{D} = \{\alpha + j\beta : -2.5 \leq \alpha \leq -0.3, -1.5 \leq \beta \leq -1.5\}$. Using the algorithm given in remark 1, we obtain $\xi = 0.003$, and the average number of iterations necessary to find a solution with a confidence $1 - \delta = 0.995$, is 1763. Table I summarize various experimentations, where $n$ is the number of iterations and $\kappa_2$ the spectral condition number. For the best result, using the random optimization algorithm (with $d = 0.025$), we obtain the matrix gain and spectral condition number presented in Table II. 

B. Example 2. 

Consider a 5-state, 3-input, 3-output pilot plant evaporator model [HO 99] given by: 

$$
\begin{align*}
A & = \begin{bmatrix}
-0.037 & 0.0123 & 0.00055 & -1.0 \\
0 & 0 & 1.0 & 0 \\
-0.37 & 0 & -0.23 & 0.0618 \\
1.25 & 0 & 0.016 & -0.0457 \\
\end{bmatrix} \\
B & = \begin{bmatrix}
0.00084 & 0.000236 \\
0 & 0 \\
0.08 & 0.804 \\
-0.0862 & -0.0665 \\
\end{bmatrix} \\
C & = \begin{bmatrix}
0 & 1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1 & 0 \\
\end{bmatrix}
\end{align*}
$$
with its open loop poles at 0, 0, −0.041, −0.051, and −1.1471. We want to find an output feedback controller \( K \), with \( |k_{ij}| \leq 5 \) \( \forall i, j \), such that the closed-loop poles are in the region defined by \( D = \{\alpha + j\beta : -1.2 \leq \alpha \leq -0.04, -0.5 \leq \beta \leq 0.5\} \). Using the algorithm given in remark 1, we obtain \( \xi = 5.13 \times 10^{-4} \), and the average number of iterations necessary to find a solution with a confidence \( 1 - \delta = 0.995 \), is 10333. Table III summarize various experimentations, where \( n \) is the number of iterations and \( \kappa_2 \) the spectral condition number. For the best result \( \kappa_2 = 18.86 \), using the random optimization algorithm (with \( d = 0.025 \)), we obtain the matrix gain and spectral condition number presented in Table IV. This spectral condition number is better than that obtained in [HO 99] which have \( \kappa_2 = 9.5 \).

VI. CONCLUSION

In this paper was presented a simple but effective method to find a robust output feedback controller via random search algorithm. The robustness of the closed-loop system is improved by the minimization of a cost function reflecting the performance of a fixed controller for a variety of plants. Two examples are presented which demonstrate the effectiveness of the proposed approach. The drawback of the proposed method is that the probability that the algorithm fails is not equal to zero for a finite number of iterations, but can be made arbitrarily small as the number of iterations increases.

REFERENCES


TABLE III

<table>
<thead>
<tr>
<th>( n )</th>
<th>Matrix gain and closed-loop poles</th>
<th>( \kappa_2 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>646</td>
<td>( K = \begin{bmatrix} -2.8917 &amp; 2.2234 &amp; 0.0715 \ -4.5221 &amp; 0.1325 &amp; 0.6331 \ -1.6834 &amp; 2.0515 &amp; -2.0068 \end{bmatrix} )</td>
<td>167.4183</td>
</tr>
<tr>
<td>634</td>
<td>( K = \begin{bmatrix} 2.7175 &amp; -1.7948 &amp; -0.2790 \ -1.7312 &amp; 4.5429 &amp; 1.1175 \ -0.8606 &amp; 1.5499 &amp; -5.2881 \end{bmatrix} )</td>
<td>677.7658</td>
</tr>
<tr>
<td>3857</td>
<td>( K = \begin{bmatrix} 0.0123 &amp; 0.9414 &amp; 0.4097 \ 0.2429 &amp; 0.0000 &amp; 0.3112 \ 0.6813 &amp; 0.3795 &amp; 0.8318 \end{bmatrix} )</td>
<td>18.8613</td>
</tr>
<tr>
<td>10084</td>
<td>( K = \begin{bmatrix} -2.8561 &amp; 1.7753 &amp; 0.2511 \ -4.2172 &amp; 1.5904 &amp; 1.5051 \ -3.3913 &amp; 0.2017 &amp; -0.4808 \end{bmatrix} )</td>
<td>62.7497</td>
</tr>
<tr>
<td>5458</td>
<td>( K = \begin{bmatrix} -0.0125 &amp; 0.1543 &amp; 0.6228 \ 0.2996 &amp; -0.3410 &amp; 0.7724 \ -4.0131 &amp; -1.6788 &amp; 0.0397 \end{bmatrix} )</td>
<td>38.0574</td>
</tr>
</tbody>
</table>

TABLE IV

<table>
<thead>
<tr>
<th>( n )</th>
<th>Matrix gain and closed-loop poles</th>
<th>( \kappa_2 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>6288</td>
<td>( K = \begin{bmatrix} -0.0135 &amp; 0.0374 &amp; 0.0870 \ 0.1409 &amp; -0.0761 &amp; 0.3660 \ -0.2131 &amp; -0.1487 &amp; 1.1119 \end{bmatrix} )</td>
<td>6.9855</td>
</tr>
</tbody>
</table>


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