A simple method to find a robust output feedback controller by random search approach

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Abstract. This paper presents a simple but effective method for finding a robust output feedback controller via a random search algorithm. The convergence of this algorithm can be guaranteed. Moreover, the probability to find a solution as well as the number of random trials can be estimated. The robustness of the closed loop system is improved by the minimization of a given cost function reflecting the performance of the controller for a set of plants. Simulation studies are used to demonstrate the effectiveness of the proposed method.

Keywords: Output feedback; Pole placement; Random search algorithm; Robustness; Condition number; Cost function; NP-hard problem.

1 Introduction

The static output feedback is an important issue not yet entirely solved (e.g., see Byrnes, 1989; Bernstein, 1992; Blondel, Gevers and Lindquist, 1995; Syrmos & al.; 1997). The problem can be stated as follows, given a linear time-invariant system, find a static output feedback so that the closed-loop system has some desirable performances. It is well known that the performances of feedback control systems are mainly determined by the locality of their closed-loop poles, it follows that a natural design approach to find a static output feedback is by means of pole placement. Compared to pole placement via state feedback, the same problem via output feedback is more complex. In fact, the static output feedback problem in the case where the feedback gains are constrained to lie in some intervals is NP-hard (Blondel and Tsitsiklis, 1997, 2000; Fu, 2004).

Starting from this negative result numerous progress has been made which modifies our notion of solving a given problem. In particular, randomized algorithms have recently received more attention in the literature. Indeed, for a randomized algorithm it is not required that it works all of the time but most of the time, in return, this kind of algorithm runs in polynomial-time (Vidyasagar, 2001). The idea of using a random algorithm to solve a complex problem is not new, and was first proposed, in the domain of automatic control, by Matyas (1965), further developments can be found in Baba (1989) and references therein, Goldberg (1989), Porter (1995), Khargonekar and Tikku (1996), Tempo, Bai and Dabbene (1997), Vidyasagar (1997), Vidyasagar (2001), Khaki-Sedigh and Bavafa-Toosi (2001), Koltchinskii & al. (2001), Abdallah & al. (2002).

In the same way as Khaki-Sedigh and Bavafa-Toosi (2001), the main objective of this paper is to present a new random approach to find a static output feedback for uncertain linear systems, which is simple and easy to use. Compared

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with the work of Khaki-Sedigh and Bavafa-Toosi (2001), the main contribution of the present paper is a mathematical justification in the use of the random search approach. In the proposed method, the probability to find a solution as well as the number of random trials can be evaluated. The robustness of the closed loop system is improved by the minimization of a given cost function reflecting the performance of the controller for a set of plants.

The paper is organized as follows. In section 2, the problem of static output feedback is formulated. Section 3 shows that the problem of regional pole placement (i.e. pole placement in a desirable domain) can be solved by an appropriate random search algorithm. The robustness issue is discussed in section 4, and section 5 presents various simulation results to demonstrate the effectiveness of this approach. Finally, section 6 concludes this paper.

2 Problem statement

Consider a multivariable linear dynamic system described by

$$\begin{cases} \dot{x}(t) = Ax(t) + Bu(t) \\ y(t) = Cx(t) \end{cases} \tag{1}$$

where $x \in \mathbf{R}^n$, $u \in \mathbf{R}^m$ and $y \in \mathbf{R}^p$ represent the state, input and output vectors, respectively, $A \in \mathbf{R}^{n\times n}$, $B \in \mathbf{R}^{n\times m}$ and $C \in \mathbf{R}^{p\times n}$ are known constant matrices. As usual, it is assumed that $\mathrm{rank}[B] = m$ and $\mathrm{rank}[C] = p$. By applying a constant output feedback law

$$u(t) = r(t) + Ky(t) \tag{2}$$

to (1), the closed-loop system is given as

$$\begin{cases} \dot{x}(t) = (A + BKC)x(t) + Br(t) \\ y(t) = Cx(t) \end{cases} \tag{3}$$

where $r \in \mathbf{R}^m$ is the reference input vector and $K \in \mathbf{R}^{m\times p}$ the output feedback gain matrix.

It was established that under the condition of (A,B) controllable and (C,A) observable and that $m + p > n$ (see Davison and Wang, 1975; Kimura, 1975) or $mp > n$ (e.g. see Wang, 1996), it exists a feedback gain matrix K such that $\lambda(A + BKC) = \Lambda$, where Λ is a given set of real and self-conjugate complex numbers, $\Lambda = \{\lambda_1,\lambda_2,\ldots,\lambda_n\}$ are the desired poles of the closed-loop system and $\lambda(M)$ is the spectrum of the square matrix M.

More precisely, $mp > n$ is a sufficient condition for the existence of a static output feedback to solve the problem of multivariable pole placement (MVPP) for the generic system, i.e. for almost all systems (Wang 1993). The condition $mp > n$ is a seminal result which is lesser conservative than $m + p > n$. Concerning this problem, further developments - including some necessary and sufficient conditions - can be found in Syrmos and Lewis (1994), Alexandridis and Paraskevopoulos (1996), Khaki-Sedigh and Bavafa-Toosi (2001).

The exact pole placement problem for output feedback is to find such a K. However, it is commonly recognized that in practical applications, the poles assigned are not required to be exactly the same as those specified. This is because the closed-loop system with poles approximately close to the desired one will possess similar desired behavior (Chu, 1993). In fact, from a practical point of view, it is sufficient to consider the pole placement in a specified stable region $\mathcal{D}$ of the complex plane, Khaki-Sedigh and Bavafa-Toosi (2001).

As shown in Blondel and Tsitsiklis (1997), the problem of finding an output feedback matrix K such that $\underline{k}_{ij} \leqslant k_{ij} \leqslant \bar{k}_{ij}\ \forall i,j$, and such that $A + BKC$ is a stable matrix is NP-hard. In a more recent work, Fu (2004) shows that the problem of pole placement via unconstrained static output feedback is also NP-hard. This implies that no efficient algorithm exists for solving such problems. In other words if a general algorithm for solving the static output feedback problem is derived, it is an exponential-time algorithm.

An alternative approach to solve this kind of problem is to use a non deterministic algorithm. The drawback of this approach is that the probability that the algorithm fails is not equal to zero for a finite number of iterations, but can be made arbitrarily small as the number of iterations increases. In return for this compromise, one hopes that the algorithm runs in polynomial time.

In the next section a random search algorithm is proposed in order to find a constrained output feedback matrix K (i.e. such that $\underline{k}_{ij} \leqslant k_{ij} \leqslant \bar{k}_{ij}\ \forall i,j$) such that $\lambda(A + BKC) \subset \mathcal{D}$, where $\mathcal{D}$ is a specified region of the complex plane determined in order to obtain a desired behavior. This problem is also NP-hard.

3 Random search algorithm approach

In this section a possible approach to solve the problem of pole placement in a desirable domain $\mathcal{D} \subset \mathbf{C}^-$ is presented. Suppose the existence of a solution, the following theorem can be used for finding a constrained output feedback matrix for the system (1).

Theorem 3.1. *If there exists an output feedback matrix K such that $\underline{k}_{ij} \leqslant k_{ij} \leqslant \bar{k}_{ij}\ \forall i,j$, and such that $\lambda(A + BKC) \subset \mathcal{D}$, with $\mathcal{D} \subset \mathbf{C}^-$, then the algorithm*

1. *Generate a $m \times p$ matrix K with random uniformly distributed elements k_{ij} on the intervals $[\underline{k}_{ij}, \bar{k}_{ij}]\ \forall i,j$.*

2. *If $\lambda(A + BKC) \not\subset \mathcal{D}$ go to step 1, otherwise stop.*

converges certainly to a solution.

Proof. Let $\mathcal{K}$ be the set of $\mathcal{D}$-stabilizing output feedback matrices K defined by

$$\mathcal{K} = \left\{K \in \mathbf{R}^{m\times p} : \lambda(A + BKC) \subset \mathcal{D}, \underline{k}_{ij} \leqslant k_{ij} \leqslant \bar{k}_{ij}\ \forall i,j\right\} \tag{4}$$

Let us consider n iterations of the algorithm, the probability so that $K \notin \mathcal{K}$ is given by the binomial probability distribution

$$P\{K \notin \mathcal{K}\} = \left[\frac{n!}{r!(n-r)!}\xi^r(1-\xi)^{n-r}\right]_{r=0} = (1-\xi)^n \tag{5}$$

where r is the number of successes (i.e. the number of times that $K \in \mathcal{K}$) and ξ the probability of elementary success. For $\xi > 0$ it is clear that $\lim_{n\to\infty}(1-\xi)^n = 0$ the algorithm then certainly converges to a solution.

Corollary 3.1. *The average number of iterations necessary to obtain a solution with a confidence at least equal to $1-\delta$ is given by*

$$n \geqslant \frac{\ln(\delta)}{\ln(1-\xi)}, \quad \text{with} \quad 0 < \xi \leqslant \frac{2\mathcal{A}(\mathcal{D}\cap\mathcal{D}_\rho)P\{\lambda(A+BKC)\subset \mathbf{C}^-\}}{\pi\left[\max\limits_K \rho(A+BKC)\right]^2} \tag{6}$$

Where $\mathbf{C}^-$ is the left half plane ($\mathbf{C}$ is the set of complex numbers), $P\{\lambda(A+BKC)\subset\mathbf{C}^-\}$ is the probability that $A+BKC$ is Hurwitz (i.e. a stable matrix), $\rho(M)$ is the spectral radius of the matrix M, that is $\rho(M) = \max(|\lambda_i|)$, with λ_i the eigenvalues of M. The quantity $\mathcal{A}(\mathcal{D}\cap\mathcal{D}_\rho)$ is the surface of the domain $\mathcal{D}\cap\mathcal{D}_\rho$, where $\mathcal{D}$ is the specified region for pole placement and $\mathcal{D}_\rho$ is the half region generated by the maximum over K of the spectral radius (see figure 1).

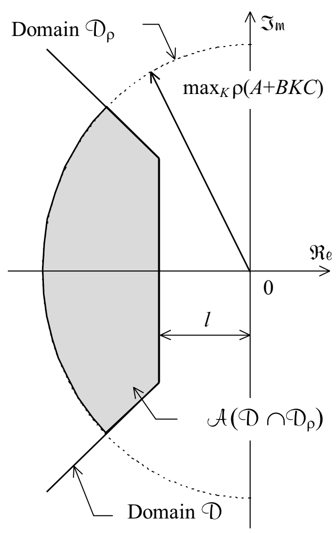

FIG. 1 – *Surface of the region $\mathcal{D}\cap\mathcal{D}_\rho$.*

Proof. From (5) we want to have $(1-\xi)^n \leqslant \delta$ which gives $n \geqslant \ln(\delta)/\ln(1-\xi)$. Suppose now that $\lambda(A_c)\subset\mathbf{C}^-$, with $A_c = A+BKC$, the probability that $K \in \mathcal{K}$

is equal to the probability that $\lambda(A_c) \subset \mathcal{D}$. Consider n random trials generating n independent identically distributed matrices K. If n goes to infinity, the ratio between the number of successes n_s and the number of trials n, is equal, by definition, to the probability $P\{\lambda(A_c) \subset \mathcal{D}/\lambda(A_c) \subset \mathbf{C}^-\}$. This probability is bounded by the ratio between the surface $\mathcal{A}(\mathcal{D} \cap \mathcal{D}_\rho)$, where $\mathcal{D}$ is the specified region for pole placement and $\mathcal{D}_\rho$ is the half region (by assumption $\lambda(A_c) \subset \mathbf{C}^-$) generated by the maximum over K of the spectral radius:

$$P\{\lambda(A_c) \subset \mathcal{D}/\lambda(A_c) \subset \mathbf{C}^-\} = \lim_{n \to \infty} \frac{n_s}{n} \leqslant \frac{2\mathcal{A}(\mathcal{D} \cap \mathcal{D}_\rho)}{\pi \rho_{max}^2} \tag{7}$$

With $\rho_{max} = \max_K \rho(A_c)$. The probability of elementary success ξ is given by $\xi = P\{(\lambda(A_c) \subset \mathbf{C}^-) \cap (\lambda(A_c) \subset \mathcal{D})\}$, by the conditional probability we have $P\{\lambda(A_c) \subset \mathcal{D}/\lambda(A_c) \subset \mathbf{C}^-\} = \xi/P\{\lambda(A_c) \subset \mathbf{C}^-\}$, which gives $\xi \leqslant 2\mathcal{A}(\mathcal{D} \cap \mathcal{D}_\rho)P\{\lambda(A_c) \subset \mathbf{C}^-\}/(\pi\rho_{max}^2)$.

Remark 3.1. The probability ξ can be estimated as relative frequency $\hat{\xi}_N = N_s/N$, where N is the total number of samples and N_s the number of samples such that $\lambda(A+BKC) \subset \mathcal{D}$. The problem is to determine the number of samples N in order to obtain a reliable probabilistic estimate. More precisely, given the accuracy $\epsilon \in [0,1]$ and the confidence $\delta \in [0,1]$, the minimum of samples N which guarantees that $P\{|\xi - \hat{\xi}_N| \leqslant \epsilon\} \geqslant 1 - \delta$ is given by the Chernoff bound (Chernoff, 1952) $N \geqslant \ln(2/\delta)/(2\epsilon^2)$. Thus, the probability ξ can be estimated using the following algorithm.

1. Choose a number of iterations N such that $N \geqslant \ln(2/\delta)/(2\epsilon^2)$.
2. Generate a $m \times p$ matrix K with random uniformly distributed elements k_{ij} on the intervals $[\underline{k}_{ij}, \bar{k}_{ij}]$ $\forall i,j$
3. If $\lambda(A + BKC) \subset \mathcal{D}$ then $N_s = N_s + 1$
4. If the number of iterations is incomplete go to step 2, otherwise stop.

The estimation of the probability ξ is then given by N_s/N. One question arises, the feasibility problem. The feasibility of pole placement by constrained output feedback is related to the spectral radius of the closed-loop state matrix. Indeed, let l be the minimal distance between the origin of the complex plane and the domain $\mathcal{D}$ of the pole placement (see figure 1). If $\max_K \rho(A + BKC) < l$, the problem is not feasible. Note that the probability $P\{(\lambda(A_c) \subset \mathbf{C}^-)\}$ as well as $\max_K \rho(A + BKC)$ can be estimated using the same approach as described in the above algorithm.

4 Robustness issue

In this section, our objective is to find an output feedback controller such that the closed-loop system remains stable for a large variety of plants. For this purpose, we consider the problem of minimal sensitivity (i.e. maximal robustness) of eigenvalues to unstructured perturbation in the system and controller parameters. An analytic solution to the problem of minimal sensitivity in static output feedback design was first given in Bavafa-Toosi and Khaki-Sedigh (2002). However, as mentioned in the above paper, the minimum achievable condition number has a lower bound (see also Kautski & al. (1985)), the problem may not have a solution. Therefore, the condition number minimization approach is usually adopted. More precisely, if an additive uncertainty Δ exists in the closed-loop system matrix, according to theorem 6 in Kautsky & al. (1985), the closed-loop state matrix $A + \Delta + BKC$ is Hurwitz if

$$\|\Delta\|_2 < \min_i \mathrm{Re}(-\lambda_i)/\kappa_2(T) \tag{8}$$

where $\|\Delta\|_2$ is the 2-norm or spectral norm of Δ, λ_i $(i = 1,2\ldots,n)$ are eigenvalues of $A + BKC$, $\kappa_2(T)$ is the spectral condition number of T, that is $\kappa_2(T) = \|T\|_2\|T^{-1}\|_2$, and T is the eigenvector matrix of $A + BKC$. From inequality (8) one can see that a smaller $\kappa_2(T)$ gives a largest bound of $\|\Delta\|_2$ and thus increases the set of plants which can be stabilized. Hence the robustness of the closed loop system can be improved by solving the following optimization problem

$$\begin{array}{ll} \text{minimize} & J = \|T\|_2\|T^{-1}\|_2 \\ \text{subject to} & K \in \mathcal{K} \end{array} \tag{9}$$

A sub-optimal solution of this optimization problem can be found using the theorem 4.1. Let us start with lemma 4.1.

Lemma 4.1. *There exists an optimal level of performance $\gamma_{min} > 1$ such that:*

$$\exists K^* \in \mathcal{K}, \quad J(K^*) = \gamma_{min} \leqslant J(K), \quad \forall K \in \mathcal{K} \tag{10}$$

There exists a bound of performance level γ_{max} such that:

$$\forall K \in \mathcal{K}, \quad J(K) \leqslant \gamma_{max} \tag{11}$$

For all levels of performance $\gamma_{min} < \gamma < \gamma_{max}$ there exists a nonempty set of solutions $\mathcal{K}_\gamma$ defined by:

$$\mathcal{K}_\gamma = \{K \in \mathcal{K} : J(K) \leqslant \gamma\} \tag{12}$$

Theorem 4.1. *For a given level of performance γ with $\gamma_{min} < \gamma < \gamma_{max}$, the random optimization algorithm*

1. *Select an initial output feedback matrix $K \in \mathcal{K}$, and a domain of exploration $[-d, d]$, $d > 0$.*

2. *Generate a $m \times p$ matrix ΔK with random uniformly distributed elements Δk_{ij} on the interval $[-d, d]$ $\forall i,j$, such that $K + \Delta K \in \mathcal{K}$.*

3. *If $J(K + \Delta K) < J(K)$ let $K = K + \Delta K$.*

4. *If $J(K) > \gamma$, go to step 2, otherwise stop.*

converges certainly to a solution $K \in \mathcal{K}_\gamma$.

Proof. Consider an initial matrix $K \in \mathcal{K}$ for which $J(K) > \gamma$. By lemma 4.1 there exists ΔK, with $K + \Delta K \in \mathcal{K}$, such that $J(K+\Delta K) < J(K)$. Consider n iterations of the algorithm, the probability that $J(K+\Delta K) > J(K)$ is given by $(1-\xi)^n$ (see the proof of theorem 3.1), where $\xi > 0$ is the probablity of success that is $Pr\{J(K+\Delta K) < J(K)\}$. It is clear that $\lim_{n\to\infty}(1-\xi)^n = 0$, then repeating the steps 2-3-4 we finally find ΔK such that $J(K+\Delta K) < J(K)$. If $J(K+\Delta K) \leqslant \gamma$ then $K + \Delta K \in \mathcal{K}_\gamma$, if not, we consider $K + \Delta K$ as a new initial matrix and repeating the reasoning above we see that the algorithm converges to an element of $\mathcal{K}_\gamma$ which is a suboptimal solution. Obviously, the optimal solution is given by the smallest level of performance γ_{min} which is unknown.

More generally, an analogue approach can be used to minimize a given cost function reflecting the performance of the controller for a given set of plants (see example 3 below).

5 Simulation results

In this section various numerical examples are presented to illustrate the validity of the proposed approach.

Example 1. Consider a 4-state, 2-input, 3-output aircraft example (Yan, Teo and Moore, 1994) given by

$$
\begin{gathered}
A = \begin{bmatrix} -0.037 & 0.0123 & 0.00055 & -1.0 \\ 0 & 0 & 1.0 & 0 \\ -6.37 & 0 & -0.23 & 0.0618 \\ 1.25 & 0 & 0.016 & -0.0457 \end{bmatrix} \\
B = \begin{bmatrix} 0.00084 & 0.000236 \\ 0 & 0 \\ 0.08 & 0.804 \\ -0.0862 & -0.0665 \end{bmatrix}, \quad C = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}
\end{gathered}
\tag{13}
$$

with its open loop poles at -0.0105, -0.2009, $-0.0507 \pm 1.1168j$. We want to find an output feedback controller K, with $|k_{ij}| \leqslant 10\ \forall i,j$, such that the closed-loop poles are in the region defined by $\mathcal{D} = \{\alpha + j\beta : -2.5 \leqslant \alpha \leqslant -0.3,\ -1.5 \leqslant \beta \leqslant -1.5\}$. Using the algorithm given in remark 1, we obtain $\xi = 0.003$, and the average number of iterations necessary to find a solution with a confidence $1-\delta = 0.995$, is 1763. The upper bound given in corollary 3.1 can be evaluated as follows. Using the same principle given in remark 3.1, the estimation of $P\{\lambda(A_c) \subset \mathbf{C}^-\}$ and $\max_K \rho(A_c)$ are given by 0.086 and 10.64 respectively. We have $\mathcal{A}(\mathcal{D}\cap\mathcal{D}_\rho) = (2.5-0.3)\times 3 = 6.6$. The upper bound of the probability ξ is then $\xi \leqslant 0.0032$. Table 1 summarizes various experimentations, where n is the number of iterations and κ_2 the spectral condition number.

n	Matrix gain and closed-loop poles	κ_2
688	$K = \begin{bmatrix} 0.2327 & 7.8452 & 3.5994 \\ -8.4385 & -6.6849 & 5.4134 \end{bmatrix}$ $\Lambda = \{-2.3981 \pm 1.3954j, -0.4670 \pm 0.6628j\}$	12.5683
1621	$K = \begin{bmatrix} 7.7624 & 8.0917 & 4.0072 \\ -8.8633 & -6.9531 & 4.4514 \end{bmatrix}$ $\Lambda = \{-0.4981 \pm 1.0898j, -2.4505 \pm 0.3201j\}$	46.7701
403	$K = \begin{bmatrix} 0.9900 & 2.1420 & 6.5898 \\ -1.0529 & -3.3637 & 2.7482 \end{bmatrix}$ $\Lambda = \{-2.4673, -0.3668 \pm 0.9934j, -0.3957\}$	8.0678
175	$K = \begin{bmatrix} 0.8021 & 3.6025 & 3.5335 \\ -7.7666 & -6.0396 & 3.0155 \end{bmatrix}$ $\Lambda = \{-2.3630 \pm 0.6832j, -0.3297 \pm 0.8411j\}$	19.8294
1501	$K = \begin{bmatrix} -0.5261 & 0.2432 & 7.2889 \\ -3.4500 & -2.6392 & -0.9264 \end{bmatrix}$ $\Lambda = \{-1.1096 \pm 1.4612j, -0.3813 \pm 0.6400j\}$	8.6107

Tab. 1. Experiment results.

For the best result, using the random optimization algorithm (with $d = 0.025$), we obtain the following matrix gain and spectral condition number.

n	Matrix gain and closed-loop poles	κ_2
75910	$K = \begin{bmatrix} 1.5474 & 7.7891 & 8.5192 \\ -1.6813 & -3.7358 & -0.4161 \end{bmatrix}$ $\Lambda = \{-2.4994, -0.3000 \pm 1.4976j, -0.3004\}$	5.2715

Tab. 2. Optimization result.

Example 2. Consider a 5-state, 3-input, 3-output pilot plant evaporator model (Ho & al., 1999) given by

$$
\begin{aligned}
A &= \begin{bmatrix} 0 & 0 & -0.0034 & 0 & 0 \\ 0 & -0.0410 & 0.0013 & 0 & 0 \\ 0 & 0 & -1.1471 & 0 & 0 \\ 0 & 0 & -0.0036 & 0 & 0 \\ 0 & 0.0940 & 0.0057 & 0 & -0.0510 \end{bmatrix} \\
B &= \begin{bmatrix} -1.0000 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0.9480 \\ 0.9160 & -1.0000 & 0 \\ -0.5980 & 0 & 0 \end{bmatrix}, \quad C = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}
\end{aligned}
\tag{14}
$$

with its open loop poles at 0, 0, -0.041, -0.051, and -1.1471. We want to find an output feedback controller K, with $|k_{ij}| \leqslant 5\ \forall i,j$, such that the closed-loop poles are in the region defined by $\mathcal{D} = \{\alpha + j\beta : -1.2 \leqslant \alpha \leqslant -0.04, -0.5 \leqslant \beta \leqslant 0.5\}$. Using the algorithm given in remark 1, we obtain $\xi = 5.13 \times 10^{-4}$, and the average number of iterations necessary to find a solution with a confidence $1-\delta = 0.995$, is 10333. The upper bound given in corollary 3.1 can be evaluated as follows. Using the same principle given in remark 3.1, the estimation of $P\{\lambda(A_c) \subset \mathbf{C}^-\}$ and $\max_K \rho(A_c)$ are given by 0.19 and 15.43 respectively. We have $\mathcal{A}(\mathcal{D}\cap\mathcal{D}_\rho) = (1.2-0.04)\times 1 = 1.16$. The upper bound of the probability ξ is then $\xi \leqslant 6.3 \times 10^{-4}$. Table 3 summarizes various experimentations, where n is the number of iterations and κ_2 the spectral condition number.

n	Matrix gain and closed-loop poles	κ_2
646	$K = \begin{bmatrix} -2.8917 & 2.2234 & 0.0715 \\ -4.5221 & 0.1325 & 0.6331 \\ -1.6834 & 2.6855 & -2.6968 \end{bmatrix}$ $\Lambda = \{-0.9769 \pm 0.3738j,\ -0.1467 \pm 0.4553j,\ -0.0537\}$	167.4183
6434	$K = \begin{bmatrix} 2.7175 & -1.7948 & -0.2750 \\ -1.7312 & 4.5429 & 1.1175 \\ -0.8666 & 1.5499 & -3.2881 \end{bmatrix}$ $\Lambda = \{-0.3659 \pm 0.4066j,\ -0.0661,\ -0.6076,\ -1.0333\}$	677.7658
3857	$K = \begin{bmatrix} 0.0123 & 0.9414 & 0.4097 \\ 0.2420 & 0.0000 & 0.3112 \\ 0.6813 & 0.3795 & 0.8318 \end{bmatrix}$ $\Lambda = \{-1.1476,\ -0.4345 \pm 0.1868j,\ -0.0669,\ -0.0403\}$	18.8613
10084	$K = \begin{bmatrix} -2.8561 & 1.7753 & 0.2511 \\ -4.2172 & 1.5954 & 1.5051 \\ -3.3913 & 0.2017 & -0.4808 \end{bmatrix}$ $\Lambda = \{-1.1092 \pm 0.0535j,\ -0.1583 \pm 0.3618j,\ -0.0466\}$	62.7497
5458	$K = \begin{bmatrix} -0.0125 & 0.1543 & 0.6328 \\ 0.2996 & -0.3410 & 0.7724 \\ -4.0131 & -1.6788 & 0.0397 \end{bmatrix}$ $\Lambda = \{-1.1353,\ -0.1324 + 0.4316j,\ -0.1272,\ -0.0513\}$	38.0574

Tab. 3. Experiment results.

For the best result $\kappa_2 = 18.86$, using the random optimization algorithm (with $d = 0.025$), we obtain the following matrix gain and spectral condition number.

n	Matrix gain and closed-loop poles	κ_2
6288	$K = \begin{bmatrix} -0.0135 & 0.0374 & 0.0870 \\ 0.1409 & -0.0761 & 0.3669 \\ -0.2131 & -0.1487 & 1.1119 \end{bmatrix}$ $\Lambda = \{-1.1435,\ -0.2865,\ -0.0422 \pm 0.0474j,\ -0.0414\}$	6.9855

Tab. 4. Optimization result.

This spectral condition number is better than that obtained in Ho & al., 1999 which have $\kappa_2 = 9.5$.

Example 3. This example concerns the design of an output dynamic feedback controller $K(s,p)$ for the longitudinal axis of an aircraft modelled by $G(s,\theta)$, where θ is the system parameters and p the controller parameters. The closed loop system is shown in figure 1, for more details see Viadyasagar (1998). The problem is to minimize the weighted sensitivity function over a set of uncertain

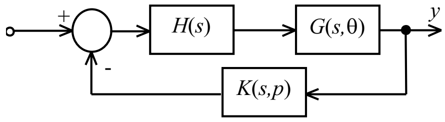

FIG. 2 – *Block diagram of the closed-loop system.*

plants, given some constraints on the nominal system. The system $G(s,\theta)$ is given in the following state space form

$$A = \begin{bmatrix} Z_\alpha & 1 - Z_q \\ M_\alpha & M_q \end{bmatrix}, \quad B = \begin{bmatrix} Z_{\delta e} \\ M_{\delta e} \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \tag{15}$$

The system parameters $\theta = [Z_\alpha \; Z_q \; M_\alpha \; M_q \; Z_{\delta e} \; M_{\delta e}]^T$ have gaussian distribution with means and standard deviations as in table 5.

Parameter	Mean (θ_0)	Standard deviation (σ)
Z_α	-0.9381	0.0736
Z_q	0.0424	0.0035
M_α	1.6630	0.1385
M_q	-0.8120	0.0676
$Z_{\delta e}$	-0.3765	0.0314
$M_{\delta e}$	-10.8791	3.4695

Tab. 5. Parameters of the aircraft model.

The transfer function $H(s)$ models the different hardware components, such as the sensor, the actuators, etc. It is given by

$$H(s) = \frac{0.000697s^2 - 0.0397s + 1}{0.000867s^2 + 0.0591s + 1} \tag{16}$$

The output dynamic feedback controllers have the following structure :

$$K(s,p) = \left[-K_a \quad - K_q \frac{1 + s\tau_1}{1 + s\tau_2} \right] \tag{17}$$

The controller parameters $p = [K_a \; K_q \; \tau_1 \; \tau_2]^T$ have uniform distributions in the ranges

$$K_a \in [0, 2], \quad K_q \in [0, 1], \quad \tau_1 \in [0.01, 0.1], \quad \tau_1 \in [0.01, 0.1] \tag{18}$$

The objective is to find the controller parameters which solve the following problem

$$\min \left\| W(I + GHK)^{-1} \right\|_\infty, \quad \text{such that} \quad \left\| \frac{0.75KG_0H}{1 + 1.25KG_0H} \right\|_\infty \leqslant 1 \tag{19}$$

where $G_0(s)$ denote the nominal system, and $W(s)$ is the weighting function given by

$$W(s) = \begin{bmatrix} \frac{2.8\times 6.28\times 31.4}{(s+6.28)(s+31.4)} & 0 \\ 0 & \frac{2.8\times 6.28\times 3.14}{(s+6.28)(s+31.4)} \end{bmatrix} \tag{20}$$

In order to solve this problem, consider the following cost function:

$$J(\theta_0,p) = \begin{cases} 1, \text{if the pair } (G_0,K) \text{ is unstable, or } \left\| \frac{0.75KG_0H}{1+1.25KG_0H} \right\|_\infty > 1 \\ \frac{\|W(I+G_0HK)^{-1}\|_\infty}{1+\|W(I+G_0HK)^{-1}\|_\infty}, \text{otherwise} \end{cases} \tag{21}$$

For a given level of nominal performance $\gamma_0 < 1$, a sub-optimal controller can be found using the following random search algorithm

1. *Select a nominal level of performance* $\gamma_0 < 1$.
2. *Generate a controller parameters p with random uniformly distributed elements on the intervals defined in (18).*
3. *If* $J(\theta_0,p) > \gamma_0$ *go to step 2, otherwise stop.*

The proof of convergence of this algorithm is similar to that of theorem 3.1. Thus we find controller parameters p_0 such that $J(\theta_0,p) \leqslant \gamma_0$. For this controller, it is crucial to verify that the worst case performance is such that $w_c(p_0) = \sup_{\theta\in\Theta} J(\theta,p_0) < 1$, where Θ is the set of more representative system parameters. For instance, for a gaussian distribution, one can choose $\Theta = [\theta_0 - 3\sigma, \theta_0 + 3\sigma]$, where θ_0 is the mean and σ the standard deviation. If there exists $\theta \in \Theta$ such that $w_c(p_0) = 1$, the controller must be rejected because we want at least stability in the worst situation. The worst case performance $w_c(p_0)$ can be estimated using $\hat{w}_c(p_0) = \sup_{\theta_i} J(\theta_i,p_0)$, where $\theta_i \in \Theta$ with $i = 1,\ldots,N$ are N i.i.d samples generated according to the probability measure P_θ on the set Θ. The number of samples necessary to have $P\{P\{w_c > \hat{w}_c\} \leqslant \epsilon\} \geqslant 1-\delta$, for a given $\epsilon \in [0,1]$ and $\delta \in [0,1]$, is such that $N \geqslant \frac{ln(1/\delta)}{\ln(1/(1-\epsilon))}$ (see Tempo, Bai and Dabbene 1997). In the same way, one can compute the average performance

$$\bar{J}_N(\theta,p_0) = \frac{1}{N}\sum_{i=1}^{N} J(\theta_i,p_0) \tag{22}$$

which reflects the performance of the controller most often obtained for a given set of plants. Obviously, the optimal controller is obtained for the smallest possible $\bar{J}(\theta,p_0)$ which is unknown but it can be approached iteratively. Table 6 summarizes successive experiments where γ_0 is the specified level of nominal performance and n the number of iterations.

γ_0	$J(\theta_0,p)$	Controller parameters	n
0.9000	0.8013	$K_a = 0.6124,\ K_q = 0.1122,\ \tau_1 = 0.0499,\ \tau_2 = 0.0520$	2
0.8000	0.7329	$K_a = 0.8874,\ K_q = 0.5925,\ \tau_1 = 0.0673,\ \tau_2 = 0.0579$	5
0.7300	0.7204	$K_a = 0.7223,\ K_q = 0.6570,\ \tau_1 = 0.0599,\ \tau_2 = 0.0450$	16
0.7200	0.7170	$K_a = 0.9046,\ K_q = 0.6471,\ \tau_1 = 0.0835,\ \tau_2 = 0.0718$	46
0.7150	0.7100	$K_a = 1.9004,\ K_q = 0.7735,\ \tau_1 = 0.0417,\ \tau_2 = 0.0107$	629

Tab. 6. Experiment results.

For the best result $J(\theta_0, p) = 0.7100$, figure 3 shows the response of the system from the initials conditions $y_1 = 1$, $y_2 = 1$ for various $\theta \in \Theta$.

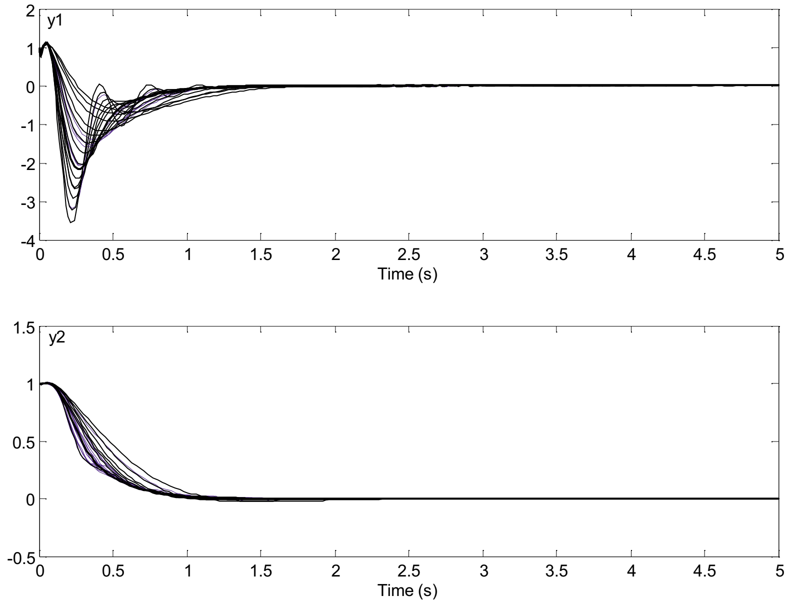

FIG. 3 – *Simulation results for various plant parameters.*

For this best result $J(\theta_0, p) = 0.7100$, the worst case performance evaluated for 70,000 plants is $\hat{w}_c = 0.7549$ and the average performance evaluated for the same number of plants is $\bar{J}_{70000}(\theta, p_0) = 0.7117$. This result is better than that obtained in Kolchinskii (2001) which obtain $\bar{J}_{66848}(\theta, p_0) = 0.7149$. In fact, for $\epsilon = 0.005$ and $\delta = 0.005$, only $N = 1057$ plants are needed to evaluate the worst case performance and the average performance. This also is a good result compared with $N = 66{,}848$ (Kolchinskii 2001).

6 Conclusion

In this paper a simple but effective method to find a robust output feedback controller via a random search algorithm was presented. The output feedback controller can be static (see examples 1 and 2) or dynamic (see example 3). The robustness of the closed-loop system is improved by the minimization of a cost function such as spectral condition number, $\mathcal{H}_2$ or $\mathcal{H}_\infty$ norms of a weighted sensitivity function and so on, reflecting the performance of a fixed controller for a variety of plants. Three examples are presented, which demonstrate the effectiveness of the proposed approach. Comparisons with the work of other authors show that the obtained results are satisfactory.

The drawback of the proposed method is that the probability that the algorithm fails is not equal to zero for a finite number of iterations, but can be made arbitrarily small as the number of iterations increases.

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