

Robust static output feedback controller synthesis using Kharitonov's theorem and Evolutionary Algorithms

R. Toscano¹, P. Lyonnet

Université de Lyon

*Laboratoire de Tribologie et de Dynamique des Systèmes CNRS UMR5513 ECL/ENISE
58 rue Jean Parot 42023 Saint-Etienne cedex 2*

Abstract. This paper presents a simple but effective tuning strategy for robust static output feedback (SOF) controllers with minimal quadratic cost in the context of multiple parametric uncertainties. Finding this type of controller is known to be computationally intractable using conventional techniques. This is mainly due to the non-convexity of the resulting control problem, which has a fixed structure. To solve this kind of control problem easily and directly, without using any complicated mathematical manipulations, we utilize Kharitonov's theorem and an evolutionary algorithm (EA) for the resolution of the underlying constrained optimization problem. Using Kharitonov's theorem, a family of bounded, robustly stable static output feedback controllers can be defined and EA is used to select the controller that ensures a minimal quadratic cost within this family. The resulting tuning strategy is applicable to both stable and unstable systems, without any limitations on the order of the process to be controlled. A numerical study was conducted to demonstrate the validity of the proposed tuning procedure.

Keywords: static output feedback (SOF), robustness, Kharitonov's theorem, non-convex optimization problems, evolutionary algorithms, quadratic cost function.

1 Introduction

In the category of non-convex problems, one important issue in control theory is the optimal synthesis of a static output feedback controller (SOF) [18, 9, 17]. Indeed, SOF controller synthesis leads to a bilinear matrix inequalities (BMI) optimization problem, which is non-convex and NP-hard to solve. However, a variety of iterative schemes to solve this type of problem have been proposed. One well-known scheme is to alternate between analysis and synthesis via linear matrix inequalities (LMI) that often results in acceptable local solutions [10, 4]. Global approaches have been also proposed to solve the BMI optimization problem [2, 11]. However these are very hard to use for a non-specialist, who often is looking for the easiest way to find a global solution to the problem. Further, these LMI/BMI based

¹E-mail address: toscano@enise.fr, Tel.: +33 477 43 84 84; Fax: +33 477 43 84 99

approaches involve the resolution of an optimization problem whose decision variables are Lyapunov variables [1]. The disadvantage of these approaches is that the number of Lyapunov variables grow quadratically with system size. Therefore, these approaches artificially introduce a large number of extra variables, whereas it is the parameters of the controller that are being sought and the controller contains a comparatively small number of unknowns. Consequently, new techniques would be useful for dealing with the non-convexity of SOF controller synthesis without introducing extra unknown variables.

In the context of multiple parametric uncertainties, the problem of designing a SOF controller has become much more difficult because the controller must stabilize a family of plants. In this framework, Kharitonov's theorem provides a test for determining the stability of a set of polynomials. This test is usually well-suited for analyzing the stability robustness of a given feedback controller, but not for synthesizing robust controllers. The few available results that exploit Kharitonov's theorem for robust synthesis of controllers are based on very restrictive models, for instance a single input single output (SISO) transfer function or state space representations in companion form. In our approach, Kharitonov's theorem is used to define a family of robustly stable SOF controllers without assuming any particular structure in the state space representation of the system. However, the problem arises of selecting the best, in some sense, robust controller within this family. This optimization problem can be solved by using the so-called evolutionary algorithms (EA) [15, 6, 5, 20, 8, 19, 13, 16]. These algorithms have demonstrated a high ability to solve non-convex optimization problems via simple stochastic strategies.

The main objective of this paper is to develop a simple and easy-to-use tuning strategy for robust SOF controllers with minimal quadratic cost, in the context of multiple parametric uncertainties. Finding this kind of controller is known to be computationally intractable using conventional techniques. Therefore, to solve this design problem easily and directly, without using any complicated mathematical manipulations, we utilize Kharitonov's theorem in association with an evolutionary algorithm (EA) to resolve the

underlying constrained optimization problem. Using Kharitonov's theorem, a family of bounded, robustly stable static output feedback controllers can be defined and EA is used to select the controller which ensures a minimal quadratic cost within this family. Since any EA can be used for this purpose, we did not limited our presentation to a particular algorithm. The resulting tuning strategy is applicable to both stable and unstable systems, without any limitation on the order of the process to be controlled. A numerical study was conducted to demonstrate the validity of the proposed tuning procedure.

2 Problem formulation

Consider a multivariable linear time-invariant (LTI) dynamic system described by:

$$\begin{cases} \dot{x}(t) = A(\theta)x(t) + B(\theta)u(t) \\ y(t) = C(\theta)x(t) \end{cases}, \quad (1)$$

where $x \in \mathbf{R}^{n_x}$, $u \in \mathbf{R}^{n_u}$ and $y \in \mathbf{R}^{n_y}$ represent the state, input, and output vectors, respectively; A , B , and C are constant matrices, with appropriate dimensions, parametrized by the system parameters $\theta \in \mathbf{R}^{n_\theta}$. As usual, it is assumed that $\text{rank}[B] = n_u$ and $\text{rank}[C] = n_y$. It is assumed that the system parameters θ (also called the parameter box) are time-invariant and lie in a bounded set Θ defined as follows:

$$\Theta = \{\theta \in \mathbf{R}^{n_\theta} : \underline{\theta} \leq_e \theta \leq_e \bar{\theta}\}, \quad (2)$$

where the notation $\leq_e$, stands for an element-by-element inequality, and the vectors $\underline{\theta} = [\underline{\theta}_1 \cdots \underline{\theta}_{n_\theta}]^T$, $\bar{\theta} = [\bar{\theta}_1 \cdots \bar{\theta}_{n_\theta}]^T$ are the bounds of the system parameters θ . With this formulation, the matrices $A(\theta)$, $B(\theta)$, and $C(\theta)$ are affected by parametric, possibly nonlinear, uncertainties. The entries of these matrices are then functions of uncertain parameters, which are bounded within intervals. In this paper, it is assumed that these entries get their extreme values from the bounds of the uncertain parameters. We suppose that the full state is not measurable and only a partial information through $y(t)$ can be used for

the control. Our main objective is to find a SOF controller that works satisfactorily for all plants parametrized on Θ . For this purpose let us consider the SOF

$$u(t) = -Ky(t), \quad (3)$$

where $u \in \mathbf{R}^m$ represents the control vector, $y \in \mathbf{R}^p$ is the measured vector, and K is the constant output feedback gain. The consideration of the SOF case is not restrictive because the dynamic output feedback case can be rephrased as a SOF control problem involving an augmented plant. Applying the output feedback (3) to (1), the closed-loop system is given by

$$\dot{x}(t) = (A(\theta) - B(\theta)KC(\theta))x(t), \quad y(t) = C(\theta)x(t). \quad (4)$$

The main objective is to solve the following optimization problem:

$$K_{opt} = \arg \min_{K \in \mathcal{K}} J(K, \theta_0), \quad (5)$$

where $J(K, \theta_0)$ is the quadratic cost of the closed-loop system for the controller K and the nominal vector of parameters θ_0 . In this paper, θ_0 represents the center of the box parameters Θ . In other words, the set Θ can be seen as a box of uncertainty around the nominal value θ_0 . In this optimization problem, $\mathcal{K}$ represents a family of bounded, robustly stable SOF controllers, i.e. a family of controllers ensuring the stability of the closed-loop system for all $\theta \in \Theta$. The resolution of (5) requires

- introducing a test that indicates if a given controller K belongs to $\mathcal{K}$. This can be done using Kahritonov's theorem (see section 3.1).
- computing the quadratic cost $J(K, \theta_0)$. This can be done by solving a Lyapunov equation (see section 3.2).
- elaborating a strategy to minimize the cost J under the constraint that $K \in \mathcal{K}$. This step can be accomplished using an evolutionary algorithm.

3 Robust synthesis of SOF controllers with minimal quadratic cost

The various requirements presented above are considered in detail in the next sections. First, we start with the problem of determining a family of bounded, robustly stable SOF controllers.

3.1 Set of bounded, robustly stable SOF controllers

For practical reasons, it is useful to limit the search space to a set of bounded SOF controllers $\mathcal{K}_B$, defined by

$$\mathcal{K}_B = \{K \in \mathbf{R}^{n_y \times n_u} : \underline{k}_{ij} \leq [K]_{i,j} \leq \bar{k}_{ij} \forall i, j\}. \quad (6)$$

The entries $[K]_{i,j}$ of the matrix K are then constrained to lie in some known interval bounded by $\underline{k}_{ij}$ and $\bar{k}_{ij}$. An element $K \in \mathcal{K}_B$ belongs to the set of robustly stable controllers if the real part of the n_x eigenvalues of the closed-loop state matrix $A_c(K, \theta) = A(\theta) - B(\theta)KC(\theta)$ is negative for all $\theta \in \Theta$. The set of bounded, robustly stable SOF controllers can then be defined as the set of controllers $K \in \mathcal{K}_B$ such that $\max_{\theta \in \Theta} \bar{\lambda}_{\Re}(A_c(K, \theta)) < 0$, where $\bar{\lambda}_{\Re}(A_c(K, \theta))$ is the largest real part of the eigenvalues of $A_c(K, \theta)$. The difficulty is that $\max_{\theta \in \Theta} \bar{\lambda}_{\Re}(A_c(K, \theta))$, cannot be evaluated because the analytic expression of $\bar{\lambda}_{\Re}(A_c(K, \theta))$ is unknown, except for small sized problems (say $n_x \leq 3$). In addition, as shown in the example presented figure 1, this function is generally non-convex and nondifferentiable and thus no efficient deterministic algorithm exists for finding the global maximum of $\bar{\lambda}_{\Re}$.

Under these conditions, we can check the robust stability by considering the extreme systems. Let $\mathcal{V}(A_c(K, \theta)) = \{A_c^1(K), \dots, A_c^l(K)\}$ be the set of $l = 2^m$ vertices of the closed-loop matrix $A_c(K, \theta)$, i.e. the set of matrices obtained by considering the bounds of the entries of $A_c(K, \theta)$, which are functions of the uncertain vector parameters θ (the

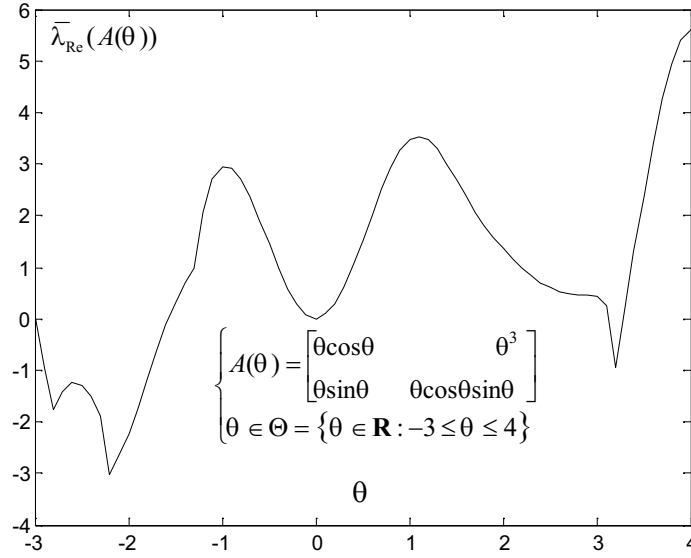


Figure 1: Non-convexity and nondifferentiability of the function $\bar{\lambda}_{\mathfrak{R}}$.

numbers of these entries are denoted by m). In other words, if $a_{ij}(K, \theta)$ represents the entry i, j of $A_c(K, \theta)$, the corresponding entries of the vertices matrices of $A_c(K, \theta)$ belong to the set

$$\left\{ \underline{a}_{ij}(K) = \min_{\theta \in \Theta} a_{ij}(K, \theta), \bar{a}_{ij}(K) = \max_{\theta \in \Theta} a_{ij}(K, \theta) \right\}. \quad (7)$$

It is then well known that the system $\dot{x}(t) = A_c(K, \theta)x(t)$ is quadratically stable, i.e. the gain matrix K is a quadratically stabilizing SOF controller for all $\theta \in \Theta$, if a symmetric positive definite matrix P can be found that satisfies the following LMI [3]:

$$(A_c^i(K))^T P + P A_c^i(K) < 0, \quad i = 1, \dots, 2^m \quad (8)$$

In fact, condition (8) implies that any matrix of the convex set

$$\mathcal{C} = \left\{ A_c(K) = \sum_{i=1}^l \alpha_i A_c^i(K), \alpha_i \geq 0, \sum_{i=1}^l \alpha_i = 1 \right\} \quad (9)$$

is Hurwitz. Since $A_c(K, \theta) \in \mathcal{C}$ for all $\theta \in \Theta$, one can conclude that the satisfaction of condition (8) implies that $A_c(K, \theta)$ is Hurwitz for all $\theta \in \Theta$. However, this condition is very conservative because condition (8) remains valid for time-varying parameters, whereas we

consider the case of uncertain linear time invariant (LTI) systems. Indeed, robust stability to time-varying parameters is more demanding than robust stability to fixed, but uncertain parameters. In addition, the number of LMI in (8) can be very large, even for a small number of uncertain parameters. For instance, if all entries of a 4×4 matrix are functions of the uncertain parameters, 2^{16} LMI must be solved. For all these reasons, the “extreme matrices” approach does not seem to be the best way to check the robust stability in the case of uncertain LTI systems with nonlinear dependence on uncertain parameters.

When the vector of uncertain parameters is time-invariant, Kharitonov’s theorem gives a simple sufficient condition for robust stability² [14].

The characteristic polynomial of the closed-loop system (4) is given by

$$\rho(s, K, \theta) = \det(sI - A_c(K, \theta)) = s^{n_x} + \rho_1(K, \theta)s^{n_x-1} + \cdots + \rho_{n_x}(K, \theta), \quad (10)$$

where the coefficients $\rho_i(K, \theta)$ ($i = 1, \dots, n_x$) are generally nonlinear functions of the elements of the feedback gain matrix K and the system parameters θ . The corresponding interval polynomial is written as follows:

$$\check{\rho}(s, K) = s^{n_x} + \check{\rho}_1(K)s^{n_x-1} + \cdots + \check{\rho}_{n_x}(K), \quad (11)$$

where $\check{\rho}_i(K)$, $i = 1, \dots, n_x$ are the intervals of the coefficients defined by

$$\check{\rho}_i(K) = [\underline{\rho}_i(K), \bar{\rho}_i(K)], \quad \text{with: } \begin{cases} \underline{\rho}_i(K) = \min_{\theta \in \mathcal{V}(\theta)} \rho_i(K, \theta) \\ \bar{\rho}_i(K) = \max_{\theta \in \mathcal{V}(\theta)} \rho_i(K, \theta) \end{cases}, \quad (12)$$

where $\mathcal{V}(\theta)$ is the set of 2^{n_θ} vertices of the parameter box Θ , defined as follows:

$$\mathcal{V}(\theta) = \left\{ \nu = [\nu_1 \cdots \nu_{n_\theta}]^T : \nu_i \in \{\underline{\theta}_i, \bar{\theta}_i\} \right\}. \quad (13)$$

Since the entries of the matrix $A_c(K, \theta)$ get their extreme values from the bounds of the uncertain parameters, this is also the case for the coefficients of the characteristic polynomial.

²When each component of the vector θ is included in only one coefficient of the characteristic polynomial $\det(sI - A_c(K, \theta))$, Kharitonov’s theorem gives the necessary and sufficient condition for robust stability. However, this decoupling condition is rarely verified in practical applications.

Given Kharitonov's theorem, an element $K \in \mathcal{K}_B$ belongs to the set of robustly stable SOF controllers if the four following polynomials are Hurwitz:

$$\left\{ \begin{array}{l} \rho_1(s, K) = \underline{\rho}_{n_x}(K) + \underline{\rho}_{n_x-1}(K)s + \bar{\rho}_{n_x-2}(K)s^2 + \bar{\rho}_{n_x-3}(K)s^3 + \\ \quad + \underline{\rho}_{n_x-4}(K)s^4 + \underline{\rho}_{n_x-5}(K)s^5 + \dots \\ \rho_2(s, K) = \underline{\rho}_{n_x}(K) + \bar{\rho}_{n_x-1}(K)s + \bar{\rho}_{n_x-2}(K)s^2 + \underline{\rho}_{n_x-3}(K)s^3 + \\ \quad + \underline{\rho}_{n_x-4}(K)s^4 + \bar{\rho}_{n_x-5}(K)s^5 + \dots \\ \rho_3(s, K) = \bar{\rho}_{n_x}(K) + \underline{\rho}_{n_x-1}(K)s + \underline{\rho}_{n_x-2}(K)s^2 + \bar{\rho}_{n_x-3}(K)s^3 + \\ \quad + \bar{\rho}_{n_x-4}(K)s^4 + \underline{\rho}_{n_x-5}(K)s^5 + \dots \\ \rho_4(s, K) = \bar{\rho}_{n_x}(K) + \bar{\rho}_{n_x-1}(K)s + \underline{\rho}_{n_x-2}(K)s^2 + \underline{\rho}_{n_x-3}(K)s^3 + \\ \quad + \bar{\rho}_{n_x-4}(K)s^4 + \bar{\rho}_{n_x-5}(K)s^5 + \dots \end{array} \right. \quad (14)$$

Therefore, a family of bounded, robustly SOF controllers can be defined as follows:

$$\mathcal{K} = \{K \in \mathcal{K}_B : \rho_i(s, K) \in \mathcal{H}, \quad i = 1, \dots, 4\}, \quad (15)$$

where $\mathcal{H}$ is the set of Hurwitz polynomials. In the case of high-order system (say $n_x \geq 4$), the computation of the closed-loop characteristic polynomial can be inextricable. Therefore, the functions $\rho_i(K, \theta)$, $i = 1, \dots, n_x$ are not available, it is then impossible to compute the bounds $\underline{\rho}_i(K)$ and $\bar{\rho}_i(K)$ analytically. However, it is possible to evaluate these bounds through an appropriate numerical procedure. For a given $K \in \mathcal{K}_B$ and a given $\theta \in \Theta$, the coefficients $\rho_i(K, \theta)$ of the closed-loop polynomial can be easily computed via the following iterative scheme [7]:

$$\begin{aligned} \rho_i(K, \theta) &= -\frac{1}{i} \text{trace}(M_i), \quad i = 1, \dots, n_x \\ M_{i+1} &= A_c(K, \theta)(M_i + \rho_i(K, \theta)I) \\ M_1 &= A_c(K, \theta). \end{aligned} \quad (16)$$

With this iterative procedure and using (12) we can then compute the bounds $\underline{\rho}_i(K)$ and $\bar{\rho}_i(K)$, for $i = 1, \dots, n_x$ exactly.

3.2 Selection of the SOF that minimizes the quadratic cost function

From a practical point of view, stability is necessary but often not sufficient. It is also very important to obtain a satisfactory performance level which can be evaluated by the mean of a given cost function. In the linear quadratic regulator problem studied here, the performance index is the standard quadratic cost function

$$J(K) = \int_0^\infty [x(t)^T Q x(t) + u(t)^T R u(t)] dt \quad (17)$$

evaluated for the nominal vector parameters θ_0 . It is thus necessary to find the robust SOF controller which minimizes the nominal quadratic cost. The weighting factors Q and R are assumed to be symmetric and definite positive matrices. For a given controller $K \in \mathcal{K}$, the corresponding quadratic cost must be determined. This can be evaluated by finding the matrix P_K that solves the following Lyapunov equation:

$$A_c(K, \theta_0)^T P_K + P_K A_c(K, \theta_0) + C(\theta_0)^T K^T R K C(\theta_0) + Q = 0. \quad (18)$$

Supposing that P_K satisfies (18) and considering the function $V(x) = x^T P_K x$, gives

$$\begin{aligned} \dot{V}(x) &= x^T (A_c(K, \theta_0)^T P_K + P_K A_c(K, \theta_0)) x \\ &= -x^T (C(\theta_0)^T K^T R K C(\theta_0) + Q) x \end{aligned} \quad (19)$$

Integrating the previous expression from $t = 0$ to ∞ and because A_c is Hurwitz, it can be deduced that

$$J(K) = \int_0^\infty [x(t)^T Q x(t) + u(t)^T R u(t)] dt = x_0^T P_K x_0, \quad (20)$$

the quadratic cost is then given by $x_0^T P_K x_0$, where x_0 is the initial condition. The dependence of $J(K)$ on the initial condition x_0 can be removed by considering x_0 as a zero-mean random variable with variance-covariance matrix $\mathbf{E}[x_0 x_0^T] = I$. Thus, the mean cost (20) can be written as $\mathbf{E}[J(K)] = \text{trace}(P_K)$.

The problem is now to determine $K_{opt} \in \mathcal{K}$, which minimizes $\mathbf{E}[J(K)]$. This problem can be formulated as follows:

$$K_{opt} = \arg \min_{K \in \mathcal{K}} \text{trace}(P_K), \quad (21)$$

where P_K is the solution of (18). This optimization problem can be solved using any EA according to the general procedure described hereafter.

Robust synthesis via EA (RSEA)

1. *Select the desired number of individuals N of the population and let $i = 1$.*
2. *Generate a sample $K_i \in \mathcal{K}_B$ according to a uniform probability distribution on $\mathcal{K}_B$.*
3. *If $K_i \notin \mathcal{K}$, go to step 2, otherwise $i = i + 1$ (the test $K_i \notin \mathcal{K}$ is performed using the procedure presented in section 3.1).*
4. *If $i < N$, go to step 2.*
5. *Compute the quadratic cost for each controller of the population (this is done by computing $\text{trace}(P_{K_i})$, $i = 1, \dots, N$, where P_{K_i} is the solution of equation (18) with $K = K_i$).*
6. *If the termination condition is satisfied, go to step 8 (the termination condition can be, for instance, a defined number of iterations).*
7. *From the results obtained step 5, generate a new population of controllers (this can be done using the usual operators of EA, i.e. selection, crossover and mutation operators), go to step 5.*
8. *The solution is given by the best candidate of the population, stop.*

4 Numerical example

In this section, the practical applicability of the proposed design method is shown on the following system:

$$\left\{ \begin{array}{l} \dot{x}(t) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & \theta_1 & \theta_2 & \theta_3 \\ \theta_4 & 0 & \theta_5 & -1 \\ \theta_4\theta_6 & \theta_7 & \theta_5\theta_6 + \theta_8 & \theta_9 - \theta_6 \end{bmatrix} x(t) + \begin{bmatrix} 0 & 0 \\ 0 & -3.91 \\ 0.035 & 0 \\ -2.53 & 0.31 \end{bmatrix} u(t) \\ y(t) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} x(t) \end{array} \right. \quad (22)$$

This system has nine unknown (but bounded) parameters which belong to following intervals:

$$\begin{aligned} \theta_1 &\in [-3.52, -2.34], & \theta_2 &\in [-5.7, -3.8], & \theta_3 &\in [0.62, 0.94] \\ \theta_4 &\in [0.07, 0.1], & \theta_5 &\in [-0.13, -0.09], & \theta_6 &\in [0.08, 0.12] \\ \theta_7 &\in [-0.05, -0.034], & \theta_8 &\in [2.1, 3.1], & \theta_9 &\in [-0.35, -0.23] \end{aligned} \quad (23)$$

these intervals define a hyperbox in the parameter space. We consider the center of this hyperbox as the nominal vector parameters:

$$\theta_0 = \frac{\theta + \bar{\theta}}{2} = [-2.93 \quad -4.75 \quad 0.78 \quad 0.085 \quad -0.11 \quad 0.1 \quad -0.042 \quad 2.6 \quad -0.29]^T. \quad (24)$$

The robust synthesis procedure RSEA described section 3.2, was applied to solve the optimization problem (21) with $R = I_2$, $Q = I_4$. The RSEA has also been implemented using a conventional genetic algorithm (see, for instance, [12]). The following parameters were used: number of generations 50, population size 50, roulette wheel selection, one point crossover with probability of 0.7 and a probability of mutation 0.07. The following controller was obtained using the RSEA:

$$K_{opt} = \begin{bmatrix} -0.3454 & 0.8515 & -1.2319 \\ -0.5861 & 0.8861 & -0.1074 \end{bmatrix} \quad (25)$$

with a nominal quadratic cost $J(K_{opt}, \theta_0) = 3.90$.

5 Conclusion

In this paper, the problem of robust synthesis of a SOF controller with minimal quadratic cost was considered in the context of multiple parametric uncertainties. This problem is known to be difficult to solve due to the non-convex nature of the underlying optimization problem. To solve this problem in a straightforward manner, we proposed a new tuning strategy that combines Kharitonov's theorem with evolutionary algorithms. Kharitonov's theorem is used to define a family of bounded, robustly stable static output feedback controllers without assuming any particular form of the model used to represent the system to be controlled. Any evolutionary algorithm can then be used to select the controller that ensures a minimal quadratic cost within this family. A numerical example demonstrated the practical applicability of the proposed design method.

References

- [1] P. Apkarian, V. Bompard, and D. Noll. Nonsmooth structured control design with application to pid loop-shaping of a process. *Int. J. Robust Nonlinear Control*, 17:1320–1342, 2007.
- [2] E. Beran, L. Vandenberghe, and S. Boyd. *A Global BMI Algorithm Based on the Generalized Benders Decomposition*. Proceedings of the European Control Conference, Bruxelles, paper no.934, 1997.
- [3] S. Boyd, L. El Ghaoui, E. Feron, and V. Balakrishnan. *Linear Matrix Inequalities in System and Control Theory*. Studies in Applied Mathematics. SIAM, Philadelphia, PA, 1994.
- [4] Y. Cao, J. Lam, and Y. Sun. Static output feedback stabilization: an ilmi approach. *Automatica*, 34(12):1641–1645, 1998.
- [5] M. Dorigo, V. Maniezzo, and A. Coloni. Ant system: Optimization by a colony of cooperating agents. *IEEE Transactions on Systems, Man, and Cybernetics-Part B*, 26(1):29–41, 1996.
- [6] R.C. Eberhart and J. Kennedy. A new optimizer using particle swarm theory. *Proceedings of the Sixth International Symposium on Micromachine and Human Science, Nagoya, Japan*, pages 39–43, 1995.
- [7] D.K. Faddeev and V.N. Faddeeva. *Computational Methods of Linear Algebra*. WH Freeman and Co., San Francisco, London, 1963.
- [8] D.B. Fogel. *Evolutionary computation: towards a new philosophy of machine intelligence, 3rd edition*. Wiley-IEEE Press, 2006.

- [9] J.C. Geromel, C.C. de Souza, and R.E. Skelton. Static output feedback controllers: Stability and convexity. *IEEE Transactions on Automatic Control*, 43(1):120–125, 1998.
- [10] L. El Ghaoui and V. Balakrishnan. Synthesis of fixed-structure controllers via numerical optimization. *Proceedings of the 33rd IEEE conference on decision and control, Lake Buena Vista (FL)*, pages 2678–2683, 1994.
- [11] K. Goh, M.G. Safonov, and G.P. Papavaissilopoulos. Global optimization for the biaffine inequality problem. *Journal of global optimization*, 7:363–380, 1995.
- [12] D. E. Goldberg. *Genetic Algorithms in Search, Optimization and Machine Learning*. Kluwer Academic Publishers, Boston, MA, 1989.
- [13] F. G. Guimarães, R. M. Palhares, F. Campelo, and H. Igarashi. Design of mixed image control systems using algorithms inspired by the immune system. *Information Sciences*, 177:4368–4386, 2007.
- [14] V. L. Kharitonov. asymptotic stability of an equilibrium position of a family of systems of differential equations. *Translated from Differentsialnye uravneniya*, 14:2086–2088, 1978.
- [15] S. Kirkpatrick, C. D. Gelatt, and M. P. Vecchi. Optimization by simulated annealing. *Science*, 220(4598):671–680, 1983.
- [16] Y. Liu, Z. Yi, H. Wu, M. Ye, and K. Chen. A tabu search approach for the minimum sum-of-squares clustering problem. *Information Sciences*, 178(12):2680–2704, 2008.
- [17] P. Makila and H.T. Toivonen. Computational methods for parametric lq problems - a survey. *IEEE Transactions on Automatic Control*, (8):658–671, 1987.
- [18] V.L. Syrmos, C.T. Abdallah, P. Dorato, and K. Grigoriadis. Static output feedback-a survey. *Automatica*, 33(2):125–137, 1997.
- [19] F. van den Bergh and A.P. Engelbrecht. A study of particle swarm optimization particle trajectories. *Information Sciences*, 176:937–971, 2006.
- [20] X. Wang, X. Z. Gao, and S. J. Ovaska. Artificial immune optimization methods and applications - a survey. *Proceedings of the IEEE International Conference On Systems, Man and Cybernetics*, 4:3415–3420, 2004.