Corrigé DS 19.12.08

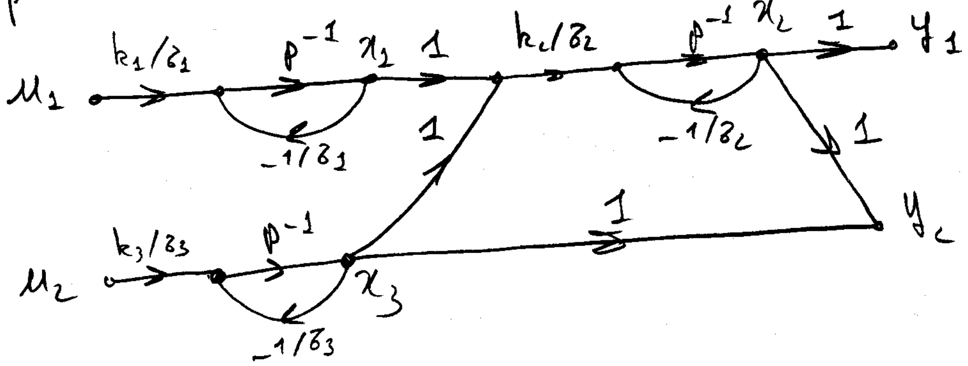
I.1. Représentation externe

$$y_1 = \frac{k_1 k_2 u_1}{(1+\tau_1 p)(1+\tau_2 p)} + \frac{k_2 k_3 u_2}{(1+\tau_2 p)(1+\tau_3 p)}$$

$$y_2 = \frac{k_1 k_2 u_1}{(1+\tau_1 p)(1+\tau_2 p)} + \left(\frac{k_3}{1+\tau_3} + \frac{k_3 k_2}{(1+\tau_2 p)(1+\tau_3 p)}\right) u_2$$

$$H(p) = \begin{bmatrix} \dfrac{k_1 k_2}{(1+\tau_1 p)(1+\tau_2 p)} & \dfrac{k_2 k_3}{(1+\tau_2 p)(1+\tau_3 p)} \\ \dfrac{k_1 k_2}{(1+\tau_1 p)(1+\tau_2 p)} & \left(1 + \dfrac{k_2}{1+\tau_2 p}\right) \dfrac{k_3}{1+\tau_3 p} \end{bmatrix}$$

I.2. Représentation d'état



$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} -1/\tau_1 & 0 & 0 \\ k_2/\tau_2 & -1/\tau_2 & k_2/\tau_2 \\ 0 & 0 & -1/\tau_3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} k_1/\tau_1 & 0 \\ 0 & 0 \\ 0 & k_3/\tau_3 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

Les pôles sont $-1/\tau_1$, $-1/\tau_2$ et $-1/\tau_3$. Le système est asymptotiquement stable si $Re(p_i) < 0 \Leftrightarrow \tau_1, \tau_2, \tau_3 > 0$.

I.3. Modèle discret :

$$\begin{cases} x(k+1) = F x(k) + G u(k) \\ y(k) = C x(k) \end{cases}$$

$$F = I + AT = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + \begin{bmatrix} -\frac{T}{\tau_1} & 0 & 0 \\ \frac{k_2 T}{\tau_2} & -\frac{T}{\tau_2} & \frac{k_2 T}{\tau_2} \\ 0 & 0 & -T/\tau_3 \end{bmatrix}$$

$$F = \begin{bmatrix} 1-T/\tau_1 & 0 & 0 \\ k_2 T/\tau_2 & 1-T/\tau_2 & k_2 T/\tau_2 \\ 0 & 0 & 1-T/\tau_3 \end{bmatrix}; \quad G = BT = \begin{bmatrix} k_1 T/\tau_1 & 0 \\ 0 & 0 \\ 0 & k_3 T/\tau_3 \end{bmatrix}$$

I.4. Modèle réduit

$$\dot{x}_3 = 0 = -\frac{1}{\tau_3} x_3 + \frac{k_3}{\tau_3} u_2 \rightarrow x_3 = k_3 u_2$$

d'où

$$\begin{cases} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -1/\tau_1 & 0 \\ k_2/\tau_2 & -\frac{1}{\tau_2} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} k_1/\tau_1 & 0 \\ 0 & \frac{k_2 k_3}{\tau_2} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \\ \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & k_3 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \end{cases}$$

I.5. Discrétisation Trapèze.

$$\begin{cases} x(k+1) = F x(k) + G u(k) \\ y(k) = C x(k) + D u(k) \end{cases}$$

$$\text{avec} \quad F = \left(I - \frac{AT}{2}\right)^{-1}\left(I + \frac{AT}{2}\right) ; \quad G = \left(I - \frac{AT}{2}\right)^{-1} BT$$

$$C = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} ; \quad D = \begin{bmatrix} 0 & 0 \\ 0 & k_3 \end{bmatrix}$$

$$I - \frac{AT}{2} = \begin{bmatrix} 1 + \frac{T}{2\tau_1} & 0 \\ -k_2 T/2\tau_2 & 1 + \frac{T}{2\tau_2} \end{bmatrix} ; \quad I + \frac{AT}{2} = \begin{bmatrix} 1 - \frac{T}{2\tau_1} & 0 \\ k_2 T/2\tau_2 & 1 - \frac{T}{2\tau_2} \end{bmatrix}$$

$$\left(I - \frac{AT}{2}\right)^{-1} = \frac{1}{\left(1+\frac{T}{2\tau_2}\right)\left(1+\frac{T}{2\tau_2}\right)} \,\text{Cofact} \begin{bmatrix} 1 + \frac{T}{2\tau_1} & -k_2 T/2\tau_2 \\ 0 & 1 + \frac{T}{2\tau_2} \end{bmatrix}$$

$$= \frac{1}{\left(1+\frac{T}{2\tau_1}\right)\left(1+\frac{T}{2\tau_2}\right)} \begin{bmatrix} 1 + T/2\tau_2 & 0 \\ \frac{k_2 T}{2\tau_2} & 1 + T/2\tau_1 \end{bmatrix}$$

$$F = \frac{1}{\left(1+\frac{T}{2\tau_1}\right)\left(1+\frac{T}{2\tau_2}\right)} \begin{bmatrix} 1 + T/2\tau_2 & 0 \\ \frac{k_2 T}{2\tau_2} & 1 + T/2\tau_1 \end{bmatrix} \begin{bmatrix} 1 - T/2\tau_1 & 0 \\ k_2 T/2\tau_2 & 1 - T/2\tau_2 \end{bmatrix}$$

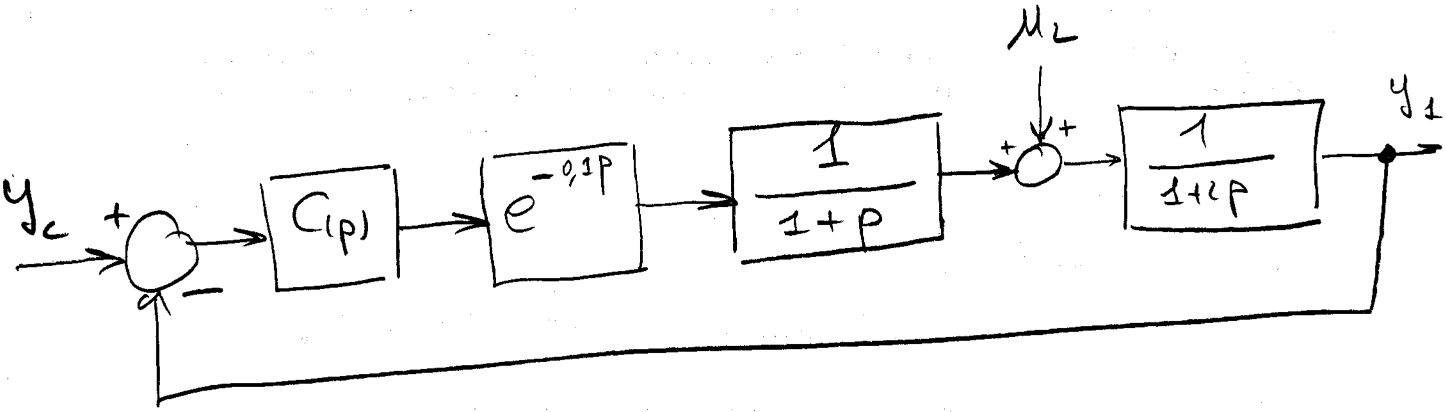
$$F = \frac{1}{\left(1+\frac{T}{2\tau_1}\right)\left(1+\frac{T}{2\tau_2}\right)} \begin{bmatrix} \left(1+\frac{T}{2\tau_2}\right)\left(1 - T/2\tau_1\right) & 0 \\ \frac{k_2 T}{2\tau_2}\left(1 - \frac{T}{2\tau_1}\right) + \frac{k_2 T}{2\tau_2}\left(1+\frac{T}{2\tau_1}\right) & \left(1+\frac{T}{2\tau_1}\right)\left(1-\frac{T}{2\tau_2}\right) \end{bmatrix} = \underbrace{\begin{bmatrix} \frac{1-\frac{T}{2\tau_1}}{1+\frac{T}{2\tau_1}} & 0 \\ \frac{k_2 T/\tau_2}{\left(1+\frac{T}{2\tau_1}\right)\left(1+\frac{T}{2\tau_2}\right)} & \frac{1-\frac{T}{2\tau_2}}{1+\frac{T}{2\tau_2}} \end{bmatrix}}_{=}$$

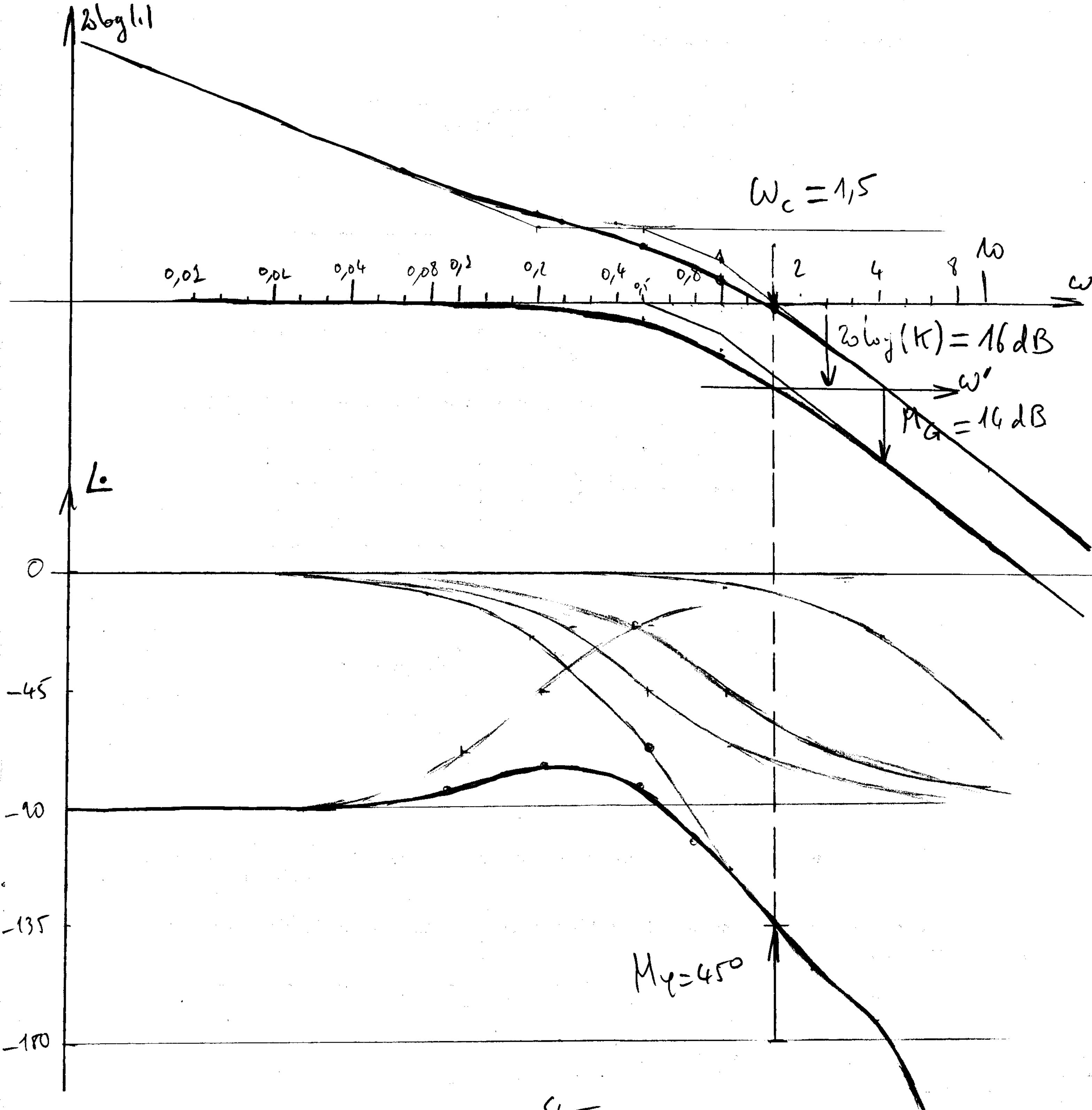
$$\boxed{\begin{bmatrix} \frac{2\tau_1 - T}{2\tau_1 + T} & 0 \\ \frac{4 k_2 \tau_1 T}{(2\tau_1+T)(2\tau_2+T)} & \frac{2\tau_2 - T}{2\tau_2 + T} \end{bmatrix}}$$

$$G = \frac{1}{\left(1+\frac{T}{2\tau_2}\right)\left(1+\frac{T}{2\tau_2}\right)} \cdot \begin{bmatrix} 1 + T/2\tau_2 & 0 \\ \frac{k_2 T}{2\tau_2} & 1 + T/2\tau_1 \end{bmatrix} \begin{bmatrix} k_1 T/\tau_1 & 0 \\ 0 & k_2 k_3 T/\tau_2 \end{bmatrix}$$

$$G = \frac{1}{\left(1+\frac{T}{2\tau_1}\right)\left(1+\frac{T}{2\tau_2}\right)} \begin{bmatrix} \frac{k_1 T}{\tau_1}\left(1+\frac{T}{2\tau_2}\right) & 0 \\ \frac{k_1 k_2 T^2}{2\tau_1\tau_2} & \frac{k_2 k_3 T}{\tau_2}\left(1+\frac{T}{2\tau_1}\right) \end{bmatrix} = \begin{bmatrix} \frac{2k_1 T}{2\tau_1 + T} & 0 \\ \frac{2k_1 k_2 T^2}{(2\tau_1+T)(2\tau_2+T)} & \frac{2k_2 k_3 T}{2\tau_2 + T} \end{bmatrix}$$

II.1. Schéma fonctionnel et diagramme de Bode.





II.2. Cas d'une action proportionnelle

On a $M\varphi = 45°$ → $20\log(K) = 16\,dB$ d'où $\underline{K \simeq 6}$

La marge de gain est alors de 14 dB et la bande passante à 0 dB de 1,5 rd/s.

II.3. Temps de réponse et période d'échantillonnage.

$$Tr \simeq 2\pi/\omega_c = \underline{4{,}2\,s.}$$

on a $Tr/Te = 21$

II.4. Correcteur P+I

$$\frac{1}{KT_i} \simeq \frac{1}{8}\,\omega_c = 0{,}1875 \to 0{,}2$$

$$20\log\left(\frac{1}{T_i}\right) \simeq 0\,dB \to \frac{1}{T_i} = 1$$

$$\underline{\begin{matrix} K = 5 \\ T_i = 1 \end{matrix}}$$

II.5. Discrétisation.

$$\frac{u(p)}{\varepsilon(p)} = \frac{1 + KT_i p}{T_i p} = \frac{1 + 5p}{p} \to \frac{du}{dt} = \varepsilon + 5\frac{d\varepsilon}{dt}$$

$$u(k+1) - u(k) = \frac{1}{5}\left(\varepsilon(k) + 5 \times 5\,(\varepsilon(k+1) - \varepsilon(k))\right)$$

$$u(k+1) = u(k) + \frac{1}{5}\varepsilon(k) + 5\,\varepsilon(k+1) - 5\,\varepsilon(k)$$

$$u(k+1) = u(k) - 4{,}8\,\varepsilon(k) + 5\,\varepsilon(k+1)$$

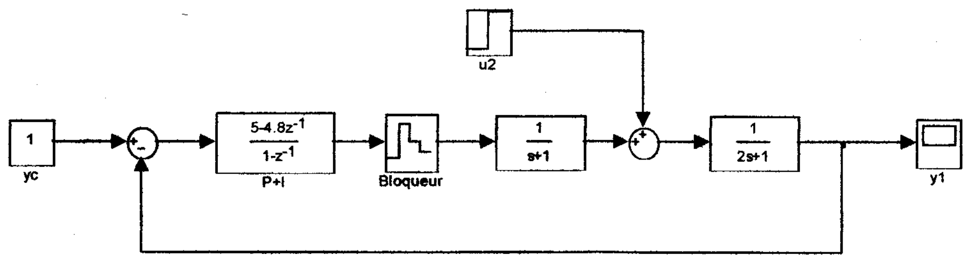
$$\underline{u(k) = u(k-1) - 4{,}8\,\varepsilon(k-1) + 5\,\varepsilon(k)}$$

Annexe simulation.

$$u(k) = u(k-1) - 4,8\,\varepsilon(k-1) + 5\,\varepsilon(k)$$

$$\rightarrow \quad \boxed{C(z) = \frac{5-4,8\,z^{-1}}{1-z^{-1}}}$$

* Schéma simulink



* Résultats :